Chapter 1

Limits and Continuity

1.1 Introduction

The notion that a *function* is a relation between two non-empty sets is a fundamental concept in understanding the rudiments of *limits* and *continuity* which are the focus of this chapter. Throughout this book, a function shall mean a rule that assigns to every element of a set, a unique corresponding element of another set. Some preliminary concepts on function will prove helpful.

1.2 Functions

Let X and Y be non-empty sets. A *function* f from X into Y is a rule that assigns to each element $x \in X$ a fully determined corresponding unique element $y \in Y$. If f is a function from X into Y, we write

$$f : X \longrightarrow Y. \tag{1.1}$$

This definition of a function f implies that every element of X is assigned, and has a corresponding unique element in Y. The set X is called the domain of the function f. If an element $x \in X$ is assigned by f to an element $y \in Y$, then y is called the *image* of x under f or the *value* of f *at* x. This is written as

$$y = f(x). \tag{1.2}$$

Whenever x is viewed as having the possibility of assuming any element of the domain X, then x is called an *independent variable* of the function f, while the corresponding image y under f is called the *dependent variable* of function f.

1.2.1 Domain

Let $f : X \longrightarrow Y$ be a function from X into Y whose value at $x \in X$ is $f(x)$, the *domain* of function f is defined as the set of all possible values of the independent variable $x \in X$ for which f is defined. Here, f is said to be defined if $f(x) \in Y$. The function f is thus *meaningful* provided its domain is known.
If domain X of function f is a subset of the set of real numbers $\mathbb{R}$, then the function f is called a function of a *real variable*, and if Y is a subset of $\mathbb{R}$, then f is called *real-valued function.*
If $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$, then f is called a *real-valued function of a real variable* and we write

$$f : X \subseteq \mathbb{R} \longrightarrow Y, \quad Y \subseteq \mathbb{R}, \tag{1.3}$$

so that $f(x) \in \mathbb{R}$ for every $x \in X \subseteq \mathbb{R}$.

Example 1.1

Determine the largest possible domains of the following functions f and g whose values at $x \in \mathbb{R}$ are

(i.) $f(x) = \dfrac{x+4}{x^2+3x+2}$ (ii.) $g(x) = \ln\left(\dfrac{1+x}{1-x}\right)$.

Solutions

(i.) Recall that domain X is the set of all possible values of the independent variable for which f is defined.
Now, given that $f(x) = \dfrac{x+4}{x^2+3x+2}$, it is clear that f is not defined whenever the denominator is zero i.e. If $x^2 + 3x + 2 = 0$ then $f(x)$ cannot be found in $\mathbb{R}$. In general division by zero is *meaningless* and should be avoided as this may lead to *contradictions*. The value of x for which denominator is zero cannot be applied to f. These values should be excluded from the domain of f.
Thus, if $x^2 + 3x + 2 \neq 0$ then $x \neq -2$ and $x \neq -1$.

Hence, the largest possible domain X of the function f defined by $f(x) = \dfrac{x+4}{x^2+3x+2}$ is

$$X = \{x \in \mathbb{R} : x \neq -2 \text{ and } x \neq -1\}.$$

(ii.) We recall that the logarithmic function $\ln(\cdot)$ is defined as a real valued function only if its argument is a positive real number i.e. $\ln u \in \mathbb{R}$

only if $u > 0$.

If $g(x) = \ln\left(\dfrac{1+x}{1-x}\right)$, then g is meaningful (defined) as a real-valued function provided

$$1 - x \neq 0 \quad \text{and} \quad \frac{1+x}{1-x} > 0.$$

This implies that $x \neq 1$, and that the inequality $\dfrac{1+x}{1-x} > 0$ should be solved for x. Clearly

$$\begin{aligned} \frac{1+x}{1-x} > 0 &\iff \frac{1+x}{(1-x)} \times \frac{1-x}{(1-x)} > 0 \\ &\implies (1+x)(1-x) > 0 \\ &\implies -1 < x < 1. \end{aligned}$$

The intersection of the sets $\{x \in \mathbb{R} : -1 < x < 1\}$ and $\{x \in \mathbb{R} : x \neq 1\}$ is

$$\{x \in \mathbb{R} : -1 < x < 1\}.$$

Hence, the domain of g is the interval $-1 < x < 1$.

Control Problem 1.2.1

Establish that the largest possible domain X of the real-valued function $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \sqrt{\frac{x}{4 - x^2}}$$

is the union of $x < -2$ and $0 \leq x < 2$.

1.2.2 Co-domain and Range

If $f : X \longrightarrow Y$ is a function from a non-empty set X into a non-empty set Y, then the set Y is called *co-domain* of function f. We recall that for any value $x \in X$, there corresponds a unique value y of f in Y. Generally, it is expected that there are some elements of Y that are not values of function f for any given $x \in X$. The set of all possible values y of function f given the domain X, is called *range* of function f. Recall that in writing $y = f(x)$, we refer to y as the dependent variable, and so the range of the function $f : X \longrightarrow Y$ can be defined as the set of values of the dependent variable

$y = f(x)$ for every $x \in X$. We shall denote the range of f as $f(X)$. It is clear that

$$f(X) \subseteq Y. \tag{1.4}$$

If $f(X) = Y$, then the function f is said to be an *onto-function.*

Example 1.2

Determine the ranges of the following real-valued functions f, g and h defined by

(i.) $f(x) = \dfrac{x+4}{x^2+3x+2}$ with largest possible domain $x \in \mathbb{R} - \{-2, -1\}$;

(ii.) $g(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ with largest possible domain $-1 < x < 1$; and

(iii.) $h(x) = \sqrt{4-x^2}$ with largest possible domain $-2 < x < 2$.

Solutions

(i.) To obtain the range of f, let $y = f(x)$ and make x subject of formula, then determine the possible values of y for which x lies in the domain. Let $y = \dfrac{x+4}{x^2+3x+2}$, then

$$yx^2 + 3yx + 2y = x + 4 \iff yx^2 + (3y-1)x + 2y - 4 = 0.$$

$$\implies x = \frac{-(3y-1) \pm \sqrt{(3y-1)^2 - 4y(2y-4)}}{2y}.$$

Clearly the quadratic equation in x has at least a real root provided the discriminant is non-negative i.e.

$$b^2 - 4ac \geq 0,$$

where $a \equiv y$, $b \equiv 3y - 1$ and $c \equiv 2y - 4$.
We have

$$(3y-1)^2 - 4y(2y-4) \geq 0 \implies y^2 + 10y + 1 \geq 0.$$

On solving this inequality we have

$$y \leq -5 - 2\sqrt{6} \quad \text{and} \quad y \geq -5 + 2\sqrt{6}.$$

Hence, if $f(x) = \dfrac{x+4}{x^2+3x+2}$, then the range of f is the union of

$$f(x) \leq -5 - 2\sqrt{6} \quad \text{and} \quad f(x) \geq -5 + 2\sqrt{6}.$$

(ii.) Let $y = g(x)$, then $y = \ln\left(\frac{1+x}{1-x}\right)$. We have

$$e^y = \frac{1+x}{1-x} \implies e^y(1-x) = 1+x.$$

Make x the subject, we have

$$x = \frac{e^y - 1}{e^y + 1}.$$

There is no real value of y for which x lies outside the domain $-1 < x < 1$. Hence, if $g(x) = \ln\left(\frac{1+x}{1-x}\right)$, then the range of g is the set of real numbers $\mathbb{R}$ which is also written as $-\infty < g(x) < \infty$.

(iii.) Let $y = h(x)$, then $y = \sqrt{4 - x^2}$. It is clear here that $y \geq 0$ and

$$y^2 = 4 - x^2 \implies x^2 + y^2 - 4 = 0.$$

Quadratic equation has at least a real root provided the discriminant is non-negative ($b^2 - 4ac \geq 0$) i.e.

$$-4(y^2 - 4) \geq 0 \implies y^2 - 4 \leq 0.$$

Solve this inequality to get $-2 \leq y \leq 2$. But it is known that y is non-negative. Taking intersection of the sets $\{y \in \mathbb{R} : -2 \leq y \leq 2\}$ and $\{y \in \mathbb{R} : y \geq 0\}$, then we have

$$\{y \in \mathbb{R} : 0 \leq 0 \leq 2\}.$$

Therefore, if $h(x) = \sqrt{4 - x^2}$, the range of h is $0 \leq h(x) \leq 2$.

Example 1.3

Determine the range of the real-valued function f defined by

$$f(x) = x^2 + 3x + 2 \quad \text{given that} \quad -4 < x < 0.$$

Solution

Let $y = f(x)$, then

$$y = x^2 + 3x + 2 \implies x^2 + 3x + 2 - y = 0.$$

Solving for x, we have

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2-y)}}{2} = \frac{-3 \pm \sqrt{1+4y}}{2}$$

That is

$$x = \frac{-3 - \sqrt{1+4y}}{2} \quad \text{or} \quad x = \frac{-3 + \sqrt{1+4y}}{2}$$

But $-4 < x < 0$, so that x lies in the intersection of the intervals $x > -4$ and $x < 0$.

Now, for $x = \dfrac{-3 - \sqrt{1+4y}}{2}$,

if $x > -4$, then

$$\frac{-3 - \sqrt{1+4y}}{2} > -4 \implies -3 - \sqrt{1+4y} > -8$$

$$\implies 5 > \sqrt{1+4y} \implies y < 6$$

If $x < 0$, then

$$\frac{-3 - \sqrt{1+4y}}{2} < 0$$

which is true for every y satisfying $1 + 4y \geq 0$. Thus, $y \geq -\frac{1}{4}$.

Taking the intersection of the intervals $y \geq -\frac{1}{4}$ and $y < 6$, we have

$$-\frac{1}{4} \leq y < 6.$$

Also, for $x = \dfrac{-3 + \sqrt{1+4y}}{2}$,

if $x > -4$, it is obvious that

$$\frac{-3 + \sqrt{1+4y}}{2} > -4 \implies \sqrt{1+4y} > -5$$

which will hold for every y satisfying $1 + 4y \geq 0$. Thus, $y \geq -\dfrac{1}{4}$.

If $x < 0$, then

$$\frac{-3 + \sqrt{1+4y}}{2} < 0 \implies \sqrt{1+4y} < 3$$

$$\implies 1 + 4y < 9 \implies y < 2$$

The intersection of the intervals $y \geq -\frac{1}{4}$ and $y < 2$ is

$$-\frac{1}{4} \leq y < 2$$

Recall that

$$x = \frac{-3 - \sqrt{1+4y}}{2} \quad \text{or} \quad x = \frac{-3 + \sqrt{1+4y}}{2}$$

so that the range of f is the union of the sets

$$\left\{y \in \mathbb{R} : -\frac{1}{4} \le y < 6\right\} \quad \text{and} \quad \left\{y \in \mathbb{R} : -\frac{1}{4} \le y < 2\right\}$$

which gives

$$-\frac{1}{4} \le f(x) < 6.$$

Control Problem 1.2.2

Determine the largest possible domain X and range $f(X)$ of the real-valued function $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{x-2}{x^2-4}.$$

Answer:

Domain is simply $X = \mathbb{R} - \{-2, +2\}$.
To obtain the range of f, observe that since $x \neq 2$, we have

$$f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}.$$

Clearly, $f(x) \neq \frac{1}{4}$ whenever $x \neq 2$.
Also, if $y = \frac{1}{x+2}$, then $x = \frac{1}{y} - 2$, so that $f(x) \neq 0$ will be required.
Hence, the range of f is $\mathbb{R} - \left\{0, \frac{1}{4}\right\}$.

Exercise 1.2

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function whose value at $x \in \mathbb{R}$ is $f(x) = x^2 + 1$ whenever $x < -2$; $f(x) = 2x - 1$ whenever $-2 \le x < 0$; and $f(x) = x + 1$ whenever $x \ge 0$. That is

$$f(x) = \begin{cases} x^2+1 & \text{if } x < -2 \\ 2x-1 & \text{if } -2 \le x < 0 \\ x+1 & \text{if } x \ge 0. \end{cases}$$

Obtain the values of (i.) $f(-3) + f(-1) + f(2)$ (ii.) $f(x-1)$.

2. Determine the largest possible domains of the following real-valued functions f, g and h whose values at a real variable x are

   (i.) $f(x) = \dfrac{x+1}{\sqrt{2x^2-3}}$ (ii.) $g(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

   (iii.) $h(x) = \tan\left(\dfrac{x-1}{x+1}\right)$.

3. Obtain the largest possible domains and ranges of the following functions f, g and h given by

   (i.) $f(x) = \dfrac{x-3}{2x+1}$ (ii.) $g(x) = \ln(x^2-4)$

   (iii.) $h(x) = 3 - 4\cos x$.

4. Determine the ranges of the following real-valued functions f and g defined by

   (i.) $f(x) = \dfrac{2+\cos 2x}{3-\cos 2x}$ for $x \in \mathbb{R}$

   (ii.) $g(x) = \sqrt{6-x-x^2}$ for $-2 < x < 1$.

5. Determine the largest possible domains and ranges of the following real-valued functions f, g and h

   (i.) $f(x) = \dfrac{x^2+3x+2}{x+4}$ (ii.) $g(x) = \dfrac{x^2+3x+2}{x^2-3x-10}$

   (iii.) $h(x) = \dfrac{x^2-9}{x-3}$.

6. Suppose $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$$

   (i.) Determine the range of f;

   (ii.) evaluate $\dfrac{f(x)-f(1)}{x-1}$ for $x \neq 1$; and

   (iii.) evaluate $\dfrac{f(x+h)-f(x)}{h}$ at $x = 1$ given that $h \neq 0$.

1.3 Some Special Functions

1.3.1 Signum Function

The *signum function* is a real-valued function whose range is the set $\{-1, 0, 1\}$. It is denoted as $\operatorname{sgn}(\cdot)$ and its value at a real value x is written as $\operatorname{sgn}(x)$. It is defined as

$$\mathrm{sgn}(x) = -1 \quad \text{if } x < 0; \quad \mathrm{sgn}(x) = 0 \quad \text{if } x = 0; \quad \text{and} \quad \mathrm{sgn}(x) = 1 \quad \text{if } x > 0.$$

That is,

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{1.5}$$

As a function of a real variable x, the largest possible domain of the signum function $\mathrm{sgn}(\cdot)$ is the set of real numbers $\mathbb{R}$. The graph of the signum function $\mathrm{sgn}(\cdot)$ is shown in figure 1.1.

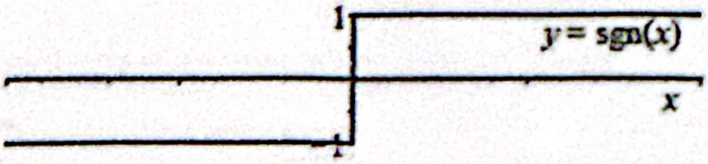

Figure 1.1: The graph of the Signum function.

Example 1.4

Let f and g be real-valued functions of a real variable x, and defined by

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = x^2 + 1.$$

Solve for x given that
(i.) $f(x) + \mathrm{sgn}(g(x)) = 1$ (ii.) $\mathrm{sgn}(f(x)) + g(x) = 1.$

Solutions

(i.) Let $u = g(x)$, then for any value of the real variable x it is obvious that $g(x) = x^2 + 1 \geq 1 > 0$ so that $u > 0$.
Following the definition of $\mathrm{sgn}(\cdot)$ that $\mathrm{sgn}(u) = 1$ for $u > 0$, we have $\mathrm{sgn}(g(x)) = 1$ so that

$$f(x) + \mathrm{sgn}(g(x)) = 1 \implies 2x - 1 + 1 = 1$$

$$x = \frac{1}{2}.$$

(ii.) Let $v = f(x)$ then

$$\operatorname{sgn}(v) = \begin{cases} -1 & \text{if } v < 0 \\ 0 & \text{if } v = 0 \\ 1 & \text{if } v > 0. \end{cases}$$

We have

$$\operatorname{sgn}(f(x)) = \begin{cases} -1 & \text{if } 2x-1 < 0 \\ 0 & \text{if } 2x-1 = 0 \\ 1 & \text{if } 2x-1 > 0 \end{cases} \implies \operatorname{sgn}(f(x)) = \begin{cases} -1 & \text{if } x < 1/2 \\ 0 & \text{if } x = 1/2 \\ 1 & \text{if } x > 1/2. \end{cases}$$

If $x < \frac{1}{2}$, then $\operatorname{sgn}(f(x)) = -1$ and we have

$$\operatorname{sgn}(f(x)) + g(x) = 1 \implies -1 + x^2 + 1 = 1$$
$$x^2 = 1 \implies x = \pm 1.$$

But since $x < \frac{1}{2}$, then $x = -1$ is a solution of $\operatorname{sgn}(f(x)) + g(x) = 1$.
Next if $x = \frac{1}{2}$, then $\operatorname{sgn}(f(x)) = 0$, and we have

$$\operatorname{sgn}(f(x)) + g(x) = 1 \implies 0 + x^2 + 1 = 1 \implies x = 0.$$

But $x = \frac{1}{2}$. Then $x = 0$ cannot be a solution of the equation $\operatorname{sgn}(f(x)) + g(x) = 1$.
Finally, if $x > \frac{1}{2}$, then $\operatorname{sgn}(f(x)) = 1$, and we have

$$\operatorname{sgn}(f(x)) + g(x) = 1 \implies 1 + x^2 + 1 = 1 \implies x^2 = -1.$$

Clearly, no real value x satisfies $x^2 = -1$.

Hence, the only solution of the equation

$$\operatorname{sgn}(f(x)) + g(x) = 1$$

is $x = -1$.

Control Problem 1.3.1

Confirm that

$$x \operatorname{sgn}(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases}$$

1.3.2 Modulus Function

The *modulus function* also called the *absolute value function* is a real-valued function whose values are non-negative. It is denoted as $|\cdot|$, and it value at a real number x is written as $|x|$. The modulus $|x|$ of a real variable x is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases} \tag{1.6}$$

This definition means $|x| = x$ whenever $x > 0$; $|x| = 0$ whenever $x = 0$; and $|x| = -x$ whenever $x < 0$. For example, $|2| = 2$ and $|-2| = -(-2) = 2$.

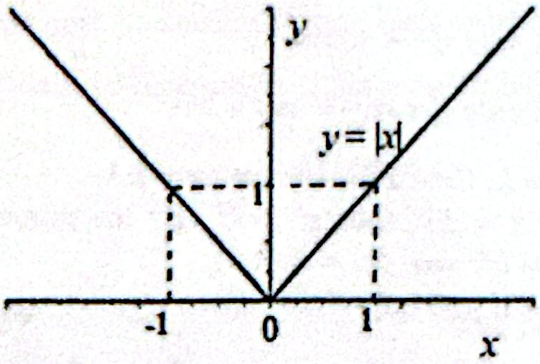

Figure 1.2: The Graph of modulus function.

The modulus $|x|$ defined in equation (1.6) can be written simply as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \tag{1.7}$$

The graph of modulus function is shown in figure 1.2. Clearly the graph is symmetric about the vertical axis.

Some Properties of the Modulus Function

Let x be a real variable, then

(a.) $|x| \geq 0$ for every $x \in \mathbb{R}$

(b.) $|x| > 0$ if and only if $x \neq 0$

(c.) $|x| = 0$ if and only if $x = 0$

(d.) $|-x| = |x|$ for every $x \in \mathbb{R}$

(e.) $|x|^2 = x^2$.

The phrase *if and only if* means *equivalent to.* For instance Property (b.) which reads $|x| > 0$ *if and only if* $x \neq 0$ means the mathematical statement $|x| > 0$ is *equivalent to* $x \neq 0$. Loosely speaking, we mean that the mathematical statements $|x| > 0$ and $x \neq 0$ convey the same meaning.
The following three theorems further identify some useful properties of the modulus function.

Theorem 1.3.1. *Let x be a real variable and k be a positive constant then*

(a.) $|x| = k$ *if and only if* $x = \pm k$,

(b.) $|x| < k$ *if and only if* $-k < x < k$,

(c.) $|x| > k$ *if and only if* $x < -k$ *and* $x > k$

Proof. (a.) If $|x| = k$, then $x^2 = k^2 \implies x = \pm k$.
Conversely, if $x = \pm k$, then $x^2 = k^2$. By the property $|x|^2 = x^2$, then we have $|x|^2 = k^2 \implies |x| = k$.
Hence, $|x| = k$ if and only if $x = \pm k$.

(b.) If $|x| < k$, then $x^2 < k^2 \implies (x+k)(x-k) < 0$, so that we have $-k < x < k$.
Conversely, if $-k < x < k$, then we have $(x+k)(x-k) < 0$
$\implies x^2 - k^2 < 0$, and we have $|x|^2 < k^2 \implies |x| < k$.

(c.) If $|x| > k$, then $x^2 > k^2 \implies (x+k)(x-k) > 0$, so that we have $x < -k$ and $x > k$.
Conversely, if $x < -k$ and $x > k$, then $(x+k)(x-k) > 0$
$\implies x^2 - k^2 > 0$, and we have $|x|^2 > k^2 \implies |x| > k$.

□

Theorem 1.3.2. *Let x and y be real numbers then*

(a.) $|xy| = |x||y|$

(b.) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ *provided* $y \neq 0$.

Proof. (a.) Clearly,

$$\begin{aligned} |xy|^2 &= (xy)^2 = x^2y^2 = |x|^2|y|^2 \\ &\implies |xy|^2 = (|x||y|)^2. \end{aligned}$$

Hence, $|xy| = |x||y|$.

(b.) To prove that

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

provided $y \neq 0$. Clearly

$$|x| = \left|\frac{x}{y} \times y\right| = \left|\frac{x}{y}\right| \times |y|$$

$$\implies \left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

□

We shall state the next theorem with proof left as exercise (See page 17).

Theorem 1.3.3 (Triangle Inequality). *Let x and y be arbitrary (any) real numbers, then*

$$|x + y| \leq |x| + |y|. \tag{1.8}$$

Remark 1.3.1

Given arbitrary (any) real numbers x, y and z, some consequences of the triangle inequality are

(a.) $|x - y| \leq |x| + |y|$

(b.) $|x| - |y| \leq |(|x| - |y|)| \leq |x - y| \leq |x| + |y|$

(c.) $|x + y + z| \leq |x| + |y| + |z|$.

Example 1.5

Describe and sketch the real-valued function f defined by

$$f(x) = |x + 1| - |x - 1|, \quad x \in \mathbb{R}.$$

Solution

Let $u = x + 1$ and $v = x - 1$, then by equation (1.7)

$$|u| = \begin{cases} u & \text{if } u \geq 0 \\ -u & \text{if } u < 0 \end{cases} \quad \text{and} \quad |v| = \begin{cases} v & \text{if } v \geq 0 \\ -v & \text{if } v < 0. \end{cases}$$

We have

$$|u| = |x+1| = \begin{cases} x + 1 & \text{if } x + 1 \geq 0 \\ -x - 1 & \text{if } x + 1 < 0 \end{cases} \implies |x+1| = \begin{cases} x + 1 & \text{if } x \geq -1 \\ -x - 1 & \text{if } x < -1 \end{cases}$$

and

$$|v| = |x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -x+1 & \text{if } x-1 < 0 \end{cases} \implies |x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -x+1 & \text{if } x < 1. \end{cases}$$

Subtracting $|x-1|$ from $|x+1|$ to get

$$|x+1| - |x-1| = \begin{cases} x+1-(x-1) & \text{if } x \geq -1 \text{ and } x \geq 1 \\ x+1-(-x+1) & \text{if } x \geq -1 \text{ and } x < 1 \\ -x-1-(x-1) & \text{if } x < -1 \text{ and } x \geq 1 \\ -x-1-(-x+1) & \text{if } x < -1 \text{ and } x < 1 \end{cases}$$

.

$$= \begin{cases} 2 & \text{if } x \geq -1 \text{ and } x \geq 1 \\ 2x & \text{if } x \geq -1 \text{ and } x < 1 \\ -2x & \text{if } x < -1 \text{ and } x \geq 1 \\ -2 & \text{if } x < -1 \text{ and } x < 1. \end{cases}$$

The intersections of intervals for each definition of f will be obtained to completely describe f. For instance, the intersection of the intervals $x \geq -1$ and $x \geq 1$ for which $f(x) = 2$ is $x \geq 1$. Also, the intersection of the intervals $x < -1$ and $x \geq 1$ for which $f(x) = -2x$ is an empty set and so this will be neglected from the definition of f. Hence f is defined by

$$f(x) = \begin{cases} 2 & \text{if } x \geq 1 \\ 2x & \text{if } -1 \leq x < 1 \\ -2 & \text{if } x < -1. \end{cases}$$

The graph of f is shown in figure 1.3.

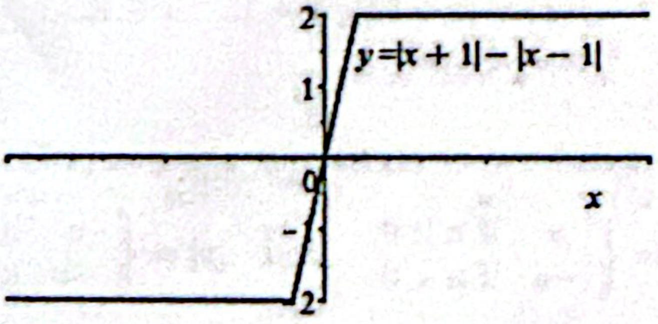

Figure 1.3: The graph of $y = |x+1| - |x-1|$.

Control Problem 1.3.2

Verify that

$$2 - \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ 3 & \text{if } x < 0. \end{cases}$$

Hence, sketch the graph of the function f defined by $f(x) = 2 - \dfrac{|x|}{x}$.

1.3.3 Greatest Integer Function

The *greatest integer function* of a real variable is a function with integers values. The greatest integer function is denoted by $[\cdot]$ and its value at a real number x is written as $[x]$ and defined as

$$[x] = k, \quad \text{where } k \text{ is greatest integer less than or equal to } x \in \mathbf{R}. \qquad (1.9)$$

For example, the value of natural base is $e = 2.7181828\cdots$, and value of π is $\pi = 3.1415\cdots$, and by definition of the greatest integer function, we have $[e] = 2$ and $[\pi] = 3$.

If the domain of the greatest integer function $[\cdot]$ is $\mathbf{R}$, then the range of $[\cdot]$ is the set of integers $\mathbf{Z}$. The graph of $[\cdot]$ is shown in figure 1.4.

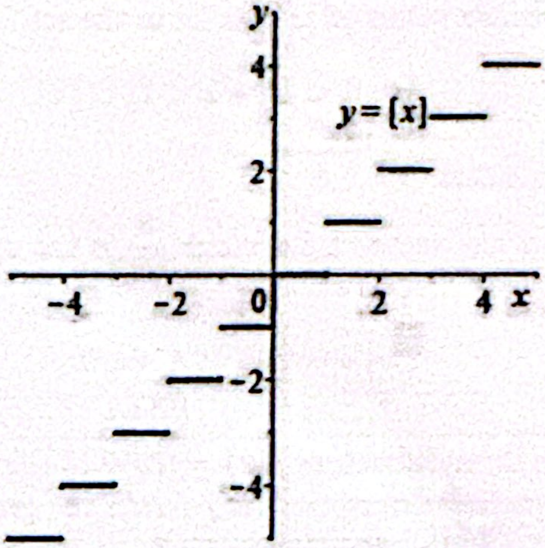

Figure 1.4: The graph of the Greatest Integer function.

Example 1.6

Given that $0 < x \leq 1$, find the possible values of
(i.) $[x] - x$ (ii.) $[x] + x$.

Solutions

(i.) Given that $0 < x \leq 1$, then by definition of $[x]$ we have

$$[x] = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly we have

$$[x] - x = \begin{cases} -x & \text{if } 0 < x < 1 \\ 1 - x & \text{if } x = 1 \end{cases} \implies [x] - x = \begin{cases} -x & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

This can be written as

$$[x] - x = \begin{cases} -x & \text{if } -1 < -x < 0 \\ 0 & \text{if } x = 1. \end{cases}$$

It is clear that the possible values of $[x] - x$ lie in $-1 < [x] - x \leq 0$.

(ii.) Clearly,

$$[x] + x = \begin{cases} 0 + x & \text{if } 0 < x < 1 \\ 1 + x & \text{if } x = 1 \end{cases} \implies [x] + x = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 & \text{if } x = 1. \end{cases}$$

Hence the possible values of $[x] + x$ lie in the set

$$\{x : 0 < [x] + x < 1\} \cup \{2\}.$$

Control Problem 1.3.3

Determine the set of real values x for which $\dfrac{x}{[x]}$ is not defined.

Answer: $0 \leq x < 1$

Exercise 1.3

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = |2x - 1| - |x + 1|$.
   (i.) Give an explicit description and sketch the graph of f
   (ii.) Determine $f(\operatorname{sgn}(x))$.

2. Determine the largest possible domain X of the real-valued function $g : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = \dfrac{5 - x^2}{[5 - x^2]}$.

3. Given that $[x] = 2$ and the real-valued function h defined by $h(x) = \dfrac{x - [x]}{x}$. Determine the largest possible domain and range of h.

4. Given that x and y are any two real numbers. Show that
   (i.) $|x+y|^2 = |x|^2 + |y|^2 + 2xy$
   (ii.) $xy \leq |x||y|$. [Hint: Note that $a \leq |a|$ for any $a \in \mathbb{R}$]
   Hence, show that

$$|x+y| \leq |x| + |y|.$$

1.4 Limits of Functions

Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function whose value at $x \in X$ is $f(x) \in \mathbb{R}$. Suppose x_o is a real number which may or may not be a member of X, and let x be a variable whose values are in X and close *as much as it is possible* to the real value x_o with $x \neq x_o$. If the values of the function f will *get* close *as much as it is desired* to a value say L, as the values of the independent variable x are sufficiently close to x_o with $x \neq x_o$, then we say L is the *limit of function* f as the variable x *approaches* x_o (or as x_o is approached by x). This is simply written as

$$L = \lim_{x \to x_o} f(x). \tag{1.10}$$

To illustrate, consider the following examples:

Example 1.7

Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{1 - \cos x}{x},$$

where x is measured in radian. Here, the largest possible domain is the set of real numbers which excludes zero. Let us investigate the values of f as variable x is close to 0.

Table 1.1: $f(x) = \dfrac{1 - \cos x}{x}$.

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\dfrac{1 - \cos x}{x}$	-0.00500	-0.00050	-0.00005	0.00005	0.00050	0.00500

From table 1.1, observe that as the values of x are getting close to 0 from the left of 0, the values of $f(x)$ get close to the value 0. Also, as values of x are getting close to 0 from the right of 0, the values of $f(x)$ get close to the value 0. Furthermore, observe that $f(x)$ will be close to the value 0 *as much as it is desired, provided* x is close to 0 *as much as it is possible,* although $x \neq 0$. Hence, the limit of $f(x)$ as x *approaches* 0 is 0. This is written as

$$L = \lim_{x \to 0} \left[\frac{1 - \cos x}{x} \right] = 0.$$

Example 1.8

Let $f : \mathbb{R} - \{3\} \longrightarrow \mathbb{R}$ be a real-valued function defined by

$$f(x) = \frac{\sin(x-3)}{x-3},$$

where the argument of $\sin(\cdot)$ is in radian. Observe that f is not defined at $x = 3$ since $f(3)$ is not a real number and in fact, meaningless.

Table 1.2: $f(x) = \dfrac{\sin(x-3)}{x-3}$.

x	2.93	2.96	2.99	3.01	3.04	3.07
$\frac{\sin(x-3)}{x-3}$	0.99918	0.99973	0.99998	0.99998	0.99973	0.99918

From table 1.2, it is clear that the values of $f(x)$ are close to 1 as values of x are getting close to 3 from both left and right hands of 3. Hence the limit of $f(x)$ as x approaches 3 is 1. This is written as

$$L = \lim_{x \to 3} \frac{\sin(x-3)}{x-3} = 1.$$

It is observed that from Examples 1.7 and 1.8, that the limit L of $f(x)$ is *unique* and *finite*. If these occur, then limit L is said to *exist*. The next example however shows that this is not always the case.

Example 1.9

Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be defined as $f(x) = \dfrac{1}{x}$.

Table 1.3: $f(x) = \dfrac{1}{x}$.

x	-10^{-2}	-10^{-3}	-10^{-4}	-10^{-5}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
$\frac{1}{x}$	-10^{2}	-10^{3}	-10^{4}	-10^{5}	10^{5}	10^{4}	10^{3}	10^{2}

From table 1.3, we observe that the absolute values of $f(x)$ *grow* large as x approaches 0, so that $f(x)$ *grows* without bound. For those values of x close to 0 but less than 0 it is obvious that $f(x) < 0$ and, $f(x)$ is large in absolute value. For those values of x close to 0 but greater than 0 it is clear that $f(x) > 0$ and $f(x)$ is large. Thus the limit of $f(x)$ as x approaches 0 is both not *unique* and *finite*, and we write

$$\lim_{x \to 0} \frac{1}{x} \quad \text{does not exist.}$$

1.4.1 Formal Definition of Limit

We recall that the modulus $|x|$ of a real variable x is a non-negative real number given by equation (1.7). Suppose $x \neq x_o$, then it is clear that $x - x_o \neq 0$. By property of modulus function, it is easy to see that

$$|x - x_o| > 0 \text{ if and only if } x - x_o \neq 0.$$

Thus, if $x - x_o \neq 0$ then $0 < |x - x_o|$.

Next, suppose L is the limit of the function f as x approaches x_o, then the value of $|f(x) - L|$ will be small *as much as it is desired*, provided $|x - x_o|$ is small *as much as it is possible.*

If $|x - x_o|$ is *small as much as it is possible*, then for any *desired* positive number ϵ, we can find a positive number δ such that $|f(x) - L|$ is smaller than the *desired* positive number ϵ, provided $0 < |x - x_o| < \delta$. Following this explanation the formal definition for the limit of a function is given as follows:

Definition 1.4.1 (Limit of a Real-Valued Function of a Real Variable). *Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function whose value at x is $f(x)$. A real number L is called the limit of $f(x)$ as x approaches x_o if given any positive number $\epsilon > 0$, there exists a positive number $\delta > 0$ depending on ϵ such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad 0 < |x - x_o| < \delta. \tag{1.11}$$

This is written as

$$L = \lim_{x \to x_o} f(x),$$

and pronounced as L is the limit of $f(x)$ as x approaches x_o. *This is also written as $f(x) \to L$ as $x \to x_o$.*

Remark 1.4.1

When evaluating $\lim_{x \to x_o} f(x)$, we keep in mind that $x \neq x_o$, but can be sufficiently close to x_o.

Example 1.10

Show that

(i.) $\lim_{x \to 3} (2x + 5) = 11$ (ii.) $\lim_{x \to 1} (x^2 + 1) = 2$ (iii.) $\lim_{x \to 2} \left(\dfrac{x+1}{x+2} \right) = 3.$

Solutions

(i.) In proving that $\lim\limits_{x\to 3}(2x+5) = 11$, one might be tempted to substitute $x = 3$ directly into the expression $2x + 5$ to get $2 \times 3 + 5 = 11$. The pitfall here is to think that $x = 3$, which will be very wrong. The right approach is to prove using the formal definition.
Let $f(x) = 2x + 3$, where $x \neq 3$, then given any $\epsilon > 0$, we want to show that it is possible to find $\delta > 0$ such that

$$|f(x) - 11| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

If $|x - 3| < \delta$, then

$$|f(x) - 11| = |2x + 5 - 11| = |2x - 6| = |2(x - 3)| = 2|x - 3| < 2\delta.$$

Now choose $\epsilon = 2\delta$ so that $\delta = \epsilon/2$, then it is clear that

$$|f(x) - 11| < 2\delta = \epsilon.$$

$\delta = \epsilon/2$ shows that for any given $\epsilon > 0$ we can find $\delta > 0$, such that

$$|f(x) - 11| < \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta = \frac{\epsilon}{2}.$$

Hence, $\lim\limits_{x\to 3}(2x + 5) = 11$.

(ii.) Let $f(x) = x^2 + 1$, where $x \neq 1$, then given any $\epsilon > 0$, we want to show that it is possible to find $\delta > 0$ such that

$$|f(x) - 2| < \epsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

From theorem (1.3.2 a), we have

$$\begin{aligned} |f(x) - 2| &= |x^2 + 1 - 2| = |x^2 - 1| \\ &= |(x + 1)(x - 1)| = |x + 1| \times |x - 1|. \end{aligned}$$

If $|x - 1| < \delta$ then it is clear that

$$|f(x) - 2| = |(x + 1)(x - 1)| = |x + 1| \times |x - 1| < |x + 1|\delta.$$

Further, by triangle inequality (theorem 1.3.3) we have

$$|x + 1| = |x - 1 + 2| \leq |x - 1| + 2 < \delta + 2.$$

Clearly,

$$|f(x) - 2| < |x + 1|\delta < (\delta + 2)\delta = \delta^2 + 2\delta.$$

Choose $\epsilon = \delta^2 + 2\delta$ and solving for δ

$$\delta = -1 \pm \sqrt{1 + \epsilon}.$$

Choosing $\delta = -1 + \sqrt{1+\epsilon}$, it is clear that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(x) - 2| < \epsilon \quad \text{for all } x \text{ satisfying} \quad 0 < |x - 1| < \delta.$$

Hence, $\lim\limits_{x \to 1} (x^2 + 1) = 2$.

(iii.) Let $f(x) = \dfrac{x+1}{x+2}$, where $x \neq 2$. We want to show that it is possible to find $\delta > 0$ such that

$$\left|f(x) - \frac{3}{4}\right| < \epsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Clearly we have

$$\left|f(x) - \frac{3}{4}\right| = \left|\frac{x+1}{x+2} - \frac{3}{4}\right| = \left|\frac{4x + 4 - 3x - 6}{4(x+2)}\right| = \frac{1}{4}\left|\frac{x-2}{x+2}\right|.$$

If $|x - 2| < \delta$, then following theorem (1.3.2 b), we have

$$\left|f(x) - \frac{3}{4}\right| = \frac{1}{4}\left|\frac{x-2}{x+2}\right| = \frac{|x-2|}{4|x+2|} < \frac{\delta}{4|x+2|}.$$

Note that

$$\begin{aligned} |x+2| &= |x - 2 + 4| \\ &= |4 - (2 - x)| \\ &\geq 4 - |2 - x| \\ &= 4 - |x - 2| > 4 - \delta. \end{aligned}$$

If $\delta < 4$ then

$$\frac{1}{|x+2|} < \frac{1}{4 - \delta}.$$

We have

$$\left|f(x) - \frac{3}{4}\right| = \frac{|x-2|}{4|x+2|} < \frac{\delta}{4(4-\delta)}.$$

Choose $\epsilon = \dfrac{\delta}{4(4-\delta)}$, then it is clear that $\delta = \dfrac{16\epsilon}{1+4\epsilon} > 0$ for any $\epsilon > 0$.
Clearly for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\left|f(x) - \frac{3}{4}\right| < \epsilon \quad \text{for all } x \text{ satisfying} \quad 0 < |x-2| < \delta.$$

Hence, $\displaystyle\lim_{x\to 2}\left(\frac{x+1}{x+2}\right) = \frac{3}{4}$.

Example 1.11

Prove that
(i.) $\displaystyle\lim_{x\to 1}(2x+1) \neq 2$ (ii.) $\displaystyle\lim_{x\to 2}\left(\frac{x-1}{x+1}\right) \neq \frac{2}{3}$.

Solutions

(i.) Let $f(x) = 2x+1$, we want to show that there is a given positive number ϵ for which we cannot find any positive number δ such that

$$|f(x) - 2| < \epsilon \quad \text{whenever} \quad 0 < |x-1| < \delta.$$

$$|f(x) - 2| = |2x + 1 - 2| = |2x - 2 + 1| = |2(x-1) + 1|.$$

Suppose $|x-1| < \delta$. By triangle inequality and the fact that $|ky| = |k| \times |y|$ we have

$$|f(x) - 2| = |2(x-1) + 1| \leq 2|x-1| + 1 < 2\delta + 1.$$

Now choose $2\delta + 1 = \epsilon$. Making δ subject of formula we have

$$\delta = \frac{\epsilon - 1}{2}.$$

Clearly, for $0 < \epsilon < 1$ we have $\delta < 0$ which contradicts the choice $|x-1| < \delta$ and we cannot find any positive number δ such that $|f(x) - 2| < \epsilon$.
Hence, $\displaystyle\lim_{x\to 1}(2x+1) \neq 2$.

(ii.) Let $f(x) = \dfrac{x-1}{x+1}$, we want to show that there is at least a given positive number ϵ for which we cannot find any positive number δ such that

$$\left|f(x) - \frac{2}{3}\right| < \epsilon \quad \text{whenever} \quad 0 < |x-2| < \delta.$$

Suppose $|x-2|<\delta$, then

$$\left|f(x)-\frac{2}{3}\right| = \left|\frac{x-1}{x+1}-\frac{2}{3}\right| = \left|\frac{3x-3-2x-2}{3(x+1)}\right| = \frac{1}{3}\left|\frac{x-5}{x+1}\right|$$

$$\implies \frac{1}{3}\left|\frac{x-5}{x+1}\right| = \frac{|x-2-3|}{3|x+1|} \leq \frac{|x-2|+3}{3|x+1|} < \frac{1}{3}\left(\frac{\delta+3}{|x-2+3|}\right).$$

Note that if $|x-2|<\delta<3$, then

$$|x-2+3| = |3-(2-x)| \geq 3-|2-x| = 3-|x-2| > 3-\delta$$

$$\implies \frac{1}{|x-2+3|} \leq \frac{1}{3-|x-2|} < \frac{1}{3-\delta}.$$

Thus, if $\delta<3$ we have

$$\left|f(x)-\frac{2}{3}\right| < \frac{1}{3}\left(\frac{\delta+3}{|x-2+3|}\right) < \frac{\delta+3}{3(3-\delta)}.$$

Now choose $\dfrac{\delta+3}{3(3-\delta)}=\epsilon$, so that by making δ the subject we have

$$\delta = \frac{9\epsilon-3}{1+3\epsilon}.$$

Since ϵ can be chosen arbitrarily, then let $0<\epsilon<1/3$, then it is clear $\delta<0$, which contradicts the choice $|x-2|<\delta$, so that no positive δ such that $\left|f(x)-\dfrac{2}{3}\right|<\epsilon$.

Hence, $\displaystyle\lim_{x\to 2}\left(\frac{x-1}{x+1}\right) \neq \frac{2}{3}$.

Control Problem 1.4.2

1. Prove that

   (i.) $\displaystyle\lim_{x\to 1}(2x+1)=3$ (ii.) $\displaystyle\lim_{x\to 2}\left(\frac{x-1}{x+2}\right)=\frac{1}{4}$

   (iii.) $\displaystyle\lim_{x\to 1}\left(\frac{x^2+3}{x+1}\right)=2.$

2. Prove that

   (i.) $\displaystyle\lim_{x\to 3}(2x+5)\neq 2$ (ii.) $\displaystyle\lim_{x\to 1}(x^2+1)\neq 1$

   (iii.) $\displaystyle\lim_{x\to 2}\left(\frac{x+1}{x+2}\right)\neq\frac{1}{2}.$

1.4.2 One-Sided Limits

Let $f : X \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be a real-valued function. It is possible to evaluate limits of f as x approaches a real number x_o from the left and right of x_o. Whenever x approaches x_o from the left, it means that $x < x_o$ while x is sufficiently close to x_o, and this is denoted as $x \to x_o^-$. Whenever x approaches x_o from the right then $x > x_o$ while x is sufficiently close to x_o, and this is denoted as $x \to x_o^+$. The limits

$$L^- = \lim_{x \to x_o^-} f(x) \quad \text{and} \quad L^+ = \lim_{x \to x_o^+} f(x)$$

are called *left-hand* and *right-hand* limits of $f(x)$ respectively, as x approaches x_o.
If L^- is the left-hand limit of $f(x)$ as x approaches x_o from the left, then $|f(x) - L^-|$ will be arbitrarily small provided x is sufficiently close to x_o and $x < x_o$. Thus, if given any positive number $\epsilon > 0$, then it is possible to find a positive number $\delta > 0$ such that if $x_0 - \delta < x < x_o$ then $|f(x) - L^-| < \epsilon$ i.e. $|f(x) - L^-| < \epsilon$ whenever $0 < x_o - x < \delta$.
Similarly, if L^+ is the right-hand limit of $f(x)$ as x approaches x_o from the right, then for any given positive number $\epsilon > 0$, it is possible to find a positive number $\delta > 0$ such that if $x_o < x < x_o + \delta$, then $|f(x) - L| < \epsilon$ i.e. $|f(x) - L^+| < \epsilon$ whenever $0 < x - x_o < \delta$. In summary, we have the following definitions for the left-hand and right-hand limits:

Definition 1.4.2. *L^- is the left-hand limit of $f(x)$ as x approaches x_o from the left if given any $\epsilon > 0$ there exists a positive number $\delta > 0$ depending on ϵ such that*

$$|f(x) - L^-| < \epsilon \quad \textit{whenever} \quad 0 < x_o - x < \delta.$$

Definition 1.4.3. *L^+ is the right-hand limit of $f(x)$ as x approaches x_o from the right if given any $\epsilon > 0$ there exists a positive number $\delta > 0$ depending on ϵ such that*

$$|f(x) - L^+| < \epsilon \quad \textit{whenever} \quad 0 < x - x_o < \delta.$$

Example 1.12

Given that a real-valued function f is defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x - 1 & \text{if } x > 1. \end{cases}$$

Show that (i.) $\lim_{x \to 1^-} f(x) = 3$ (ii.) $\lim_{x \to 1+} f(x) = 1$.

Solutions

(i.) Given any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - 3| < \epsilon \quad \text{whenever} \quad 0 < 1 - x < \delta.$$

Suppose $1 - x < \delta$. Since $x \to 1^-$, then $x < 1$, and $f(x) = 2x + 1$. Clearly,

$$|f(x) - 3| = |2x + 1 - 3| = |2x - 2| = 2|x - 1|.$$

By definition of the modulus, we note that

$$|x-1| = \begin{cases} x - 1 & \text{if } x - 1 \geq 0 \\ -(x-1) & \text{if } x - 1 < 0 \end{cases} \iff |x-1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$$

so that if $x < 1$ then $|x - 1| = 1 - x$ and we have

$$|f(x) - 3| = 2|x - 1| = 2(1 - x) < 2\delta.$$

Choose $\epsilon = 2\delta$, then it is clear that $\delta = \epsilon/2$ and $|f(x) - 3| < \epsilon$. Hence, since given any $\epsilon > 0$ we can find $\delta = \epsilon/2 > 0$, for which $|f(x) - 3| < \epsilon$ whenever $0 < 1 - x < \delta$, then $\lim\limits_{x \to 1^-} f(x) = 3$.

(ii.) To show that for any given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x) - 1| < \epsilon \quad \text{whenever} \quad 0 < x - 1 < \delta.$$

Suppose $x - 1 < \delta$. Since $x \to 1^+$, then $x > 1$, so that $f(x) = 2x - 1$. Clearly,

$$|f(x) - 1| = |2x - 1 - 1| = |2x - 2| = 2|x - 1|.$$

By the definition of the modulus, it is clear that if $x > 1$ then $|x - 1| = x - 1$ and we have

$$|f(x) - 1| = 2|x - 1| = 2(x - 1) < 2\delta.$$

Choose $\epsilon = 2\delta$, then it is clear that $\delta = \epsilon/2$ and $|f(x) - 1| < \epsilon$. Hence, $\lim\limits_{x \to 1^+} f(x) = 1$.

Control Problem 1.4.3

1. Let the real-valued function f be defined by

$$f(x) = \begin{cases} 3x + 5 & \text{if } x > 2 \\ 2x + 7 & \text{if } x < 2. \end{cases}$$

   Prove that $\lim\limits_{x \to 2^+} f(x) = 11 = \lim\limits_{x \to 2^-} f(x)$.

2. Prove that

$$\lim_{x \to -5^-} |x + 5| = 0 = \lim_{x \to -5^+} |x + 5|.$$

1.4.3 Existence of a Limit

Definition 1.4.4. *Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function whose value at $x \in X$ is $f(x)$. The limit of f as x approaches real number x_o is said to exist if and only if the left-hand and right-hand limits are equal i.e.*

$$\lim_{x\to x_o^-} f(x) = \lim_{x\to x_o^+} f(x).$$

If the left-hand and right-hand limits are equal then

$$\lim_{x\to x_o} f(x) = \lim_{x\to x_o^-} f(x) = \lim_{x\to x_o^+} f(x). \tag{1.12}$$

Remark 1.4.2

If $\lim_{x\to x_o^-} f(x) \neq \lim_{x\to x_o^+} f(x)$, then it is said that $\lim_{x\to x_o} f(x)$ does not exist.

Example 1.13

Discuss the existence of $\lim_{x\to 0} f(x)$ given that
(i.) $f(x) = \text{sgn}(x)$ (ii.) $f(x) = [x]$ (iii.) $f(x) = |x|$.

Solutions

(i.) Recall that

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

If $x < 0$ then $f(x) = -1$, thus as x approaches 0 from the left, it is clear that $f(x)$ does not change from -1 and we have $\lim_{x\to 0^-} f(x) = -1$.
If $x > 0$, we have $f(x) = 1$ and thus as x approaches 0 from the right, it is clear that $f(x)$ does not change from 1 and we have $\lim_{x\to 0^+} f(x) = 1$.
Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Hence $\lim_{x\to 0} f(x)$ does not exist.

(ii.) $f(x) = [x]$ is the greatest integer less than or equal to x, where x is a real variable.
If $x < 0$ and close to 0 such that $-1 < x < 0$, then $f(x) = [x] = -1$.
Thus, $\lim_{x\to 0^-} f(x) = -1$.
Also, if $x > 0$ and close to 0 such that $0 < x < 1$, then $f(x) = [x] = 0$.
Thus, $\lim_{x\to 0^+} f(x) = 0$.
Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Hence, $\lim_{x\to 0} f(x)$ does not exist.

(iii.) Recall that

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

If $x < 0$, then $f(x) = -x$, so that $f(x)$ approaches 0 as x approaches 0 from the left i.e. $\lim_{x \to 0^-} f(x) = 0$.

If $x > 0$, then $f(x) = x$, so that $f(x)$ approaches 0 as x approaches 0 from the right i.e. $\lim_{x \to 0^+} f(x) = 0$.

Clearly, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$. Hence, $\lim_{x \to 0} f(x)$ exists and is equal to 0 i.e. $\lim_{x \to 0} f(x) = 0$.

Control Problem 1.4.3

Investigate the existence of the following limits

(i.) $\lim_{x \to 0} \operatorname{sgn}(x^2)$ (ii.) $\lim_{x \to 0} [x^2]$ (iii.) $\lim_{x \to 0} |x^2|$.

1.4.4 Some Theorems on Limits

The following theorems will be useful in evaluating limits of functions. We shall assume that f and g are real valued functions of one real variable and x_o a real number.

Theorem 1.4.1. *Let k be a real constant independent of x, then*

$$\lim_{x \to x_o} k = k. \tag{1.13}$$

Proof. Let $f(x) = k$. Given any $\epsilon > 0$, we show that there is a $\delta > 0$ such that

$$|f(x) - k| < \epsilon \quad \text{whenever} \quad 0 < |x - x_o| < \delta.$$

If $|x - x_o| < \delta$, then

$$\begin{aligned} |f(x) - k| &= |k - k| = 0 = |x - x_o - (x - x_o)| \\ &\leq |x - x_o| + |x - x_o| < 2\delta = \epsilon, \end{aligned}$$

where $\delta = \epsilon/2$ so that for any $\epsilon > 0$, we have $\delta > 0$. Hence, $\lim_{x \to x_o} k = k$. ☐

For example, $\lim_{x \to 2} 5 = 5$.

Theorem 1.4.2. *Let M be a real constant, and $\lim_{x \to x_o} f(x) = L$, then*

$$\lim_{x \to x_o} Mf(x) = M \lim_{x \to x_o} f(x) = ML. \tag{1.14}$$

Proof. If $\lim\limits_{x\to x_o} f(x) = L$, then for any given $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - x_o| < \delta$.
Now we show that for any $\epsilon^* > 0$, it is possible to find $\delta^* > 0$ such that

$$|Mf(x) - ML| < \epsilon^* \quad \text{whenever} \quad 0 < |x - x_o| < \delta^*.$$

Clearly,

$$|Mf(x) - ML| = |M(f(x) - L)| = |M| \times |f(x) - L| < |M|\epsilon$$

whenever $0 < |x - x_o| < \delta$. Choose $\epsilon^* = |M|\epsilon$ and $\delta^* = \delta$ so that for any $\epsilon^* > 0$, it is possible to find $\delta^* = \delta > 0$ such that

$$|Mf(x) - ML| < \epsilon^* \quad \text{whenever} \quad 0 < |x - x_o| < \delta^*.$$

Hence, $\lim\limits_{x\to x_o} Mf(x) = ML$. □

Theorem 1.4.3 (Limit of Sum). *Given that* $\lim\limits_{x\to x_o} f(x) = L$ *and* $\lim\limits_{x\to x_o} g(x) = N$, *then*

$$\lim_{x\to x_o} (f(x) + g(x)) = \lim_{x\to x_o} f(x) + \lim_{x\to x_o} g(x) = L + N. \tag{1.15}$$

Proof. If $\lim\limits_{x\to x_o} f(x) = L$ and $\lim\limits_{x\to x_o} g(x) = N$, then given any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ it is possible to find $\delta_1 > 0$ depending ϵ_1 and $\delta_2 > 0$ depending on ϵ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{whenever} \quad 0 < |x - x_o| < \delta_1 \tag{1.16}$$

and

$$|g(x) - N| < \epsilon_2 \quad \text{whenever} \quad 0 < |x - x_o| < \delta_2. \tag{1.17}$$

Clearly,

$$\begin{aligned}|f(x) + g(x) - (L + N)| &= |f(x) - L + g(x) - N| \\ &\leq |f(x) - L| + |g(x) - N| < \epsilon_1 + \epsilon_2\end{aligned}$$

whenever

$$0 < |x - x_o| < \delta_1 \quad \text{and} \quad 0 < |x - x_o| < \delta_2. \tag{1.18}$$

Take intersection of the two intervals in (1.18), and choose minimum of δ_1 and δ_2 as δ i.e. $\delta = \min(\delta_1, \delta_2)$. If $\epsilon = \epsilon_1 + \epsilon_2$, then since ϵ_1 and ϵ_2 can be chosen arbitrarily, then ϵ is also an arbitrary (any) positive number, and thus we have

$$|f(x) + g(x) - (L + N)| < \epsilon \quad \text{for all } x \text{ satisfying} \quad 0 < |x - x_o| < \delta.$$

Hence, $\lim\limits_{x\to x_o} (f(x) + g(x)) = \lim\limits_{x\to x_o} f(x) + \lim\limits_{x\to x_o} g(x)$. □

Remark 1.4.3

(a.) From theorem 1.4.2 it is clear that $\lim_{x\to x_o}(-g(x)) = -\lim_{x\to x_o} g(x)$, then by theorem 1.4.3, we have

$$\lim_{x\to x_o}(f(x)-g(x)) = \lim_{x\to x_o}(f(x)+(-g(x))) = \lim_{x\to x_o} f(x) - \lim_{x\to x_o} g(x). \tag{1.19}$$

(b.) Suppose $f_1, f_2, \cdots, f_n$ are $n-$real-valued functions of a real variable x, and

$$\lim_{x\to x_o} f_1(x),\ \lim_{x\to x_o} f_2(x), \cdots, \lim_{x\to x_o} f_n(x) \quad \text{exist,}$$

then

$$\begin{aligned}\lim_{x\to x_o}&(f_1(x)+f_2(x)+\cdots+f_n(x))\\ &= \lim_{x\to x_o} f_1(x) + \lim_{x\to x_o} f_2(x) + \cdots + \lim_{x\to x_o} f_n(x).\end{aligned} \tag{1.20}$$

This can be written using the notation for sum as

$$\lim_{x\to x_o}\left(\sum_{k=1}^{n} f_k(x)\right) = \sum_{k=1}^{n}\left(\lim_{x\to x_o} f_k(x)\right). \tag{1.21}$$

That is, *limit of sum is sum of limits.*

Theorem 1.4.4 (Limit of Product). *Given that* $\lim_{x\to x_o} f(x) = L$ *and* $\lim_{x\to x_o} g(x) = N$, *then*

$$\lim_{x\to x_o} f(x)g(x) = \lim_{x\to x_o} f(x) \lim_{x\to x_o} g(x) = LN, \tag{1.22}$$

Proof. Given two arbitrary positive numbers ϵ_1 and ϵ_2, it is possible to find δ_1 and δ_2 such that statements in equations (1.16) and (1.17) are satisfied. Clearly,

$$\begin{aligned}|f(x)g(x)-LN| &= |f(x)g(x)-Lg(x)+Lg(x)-LN|\\ &= |g(x)(f(x)-L)+L(g(x)-N)|\\ &\le |g(x)(f(x)-L)|+|L(g(x)-N)|\\ &= |g(x)|\cdot|f(x)-L|+|L|\cdot|g(x)-N|\\ &< |g(x)|\epsilon_1+|L|\epsilon_2\\ &= |g(x)-N+N|\epsilon_1+|L|\epsilon_2\\ &\le |g(x)-N|\epsilon_1+|N|\epsilon_1+|L|\epsilon_2\\ &< \epsilon_1\epsilon_2+|N|\epsilon_1+|L|\epsilon_2.\end{aligned}$$

Choose $\epsilon = \epsilon_1\epsilon_2 + |N|\epsilon_1 + |L|\epsilon_2$, then ϵ can be arbitrarily chosen for arbitrary values of ϵ_1 and ϵ_2 whenever, $0 < |x - x_o| < \delta$, where $\delta = \min(\delta_1, \delta_2)$. It is possible to find $\delta > 0$ for any given $\epsilon > 0$ such that

$$|f(x)g(x) - LN| < \epsilon \quad \text{whenever} \quad 0 < |x - x_o| < \delta.$$

Hence, $\lim\limits_{x\to x_o} f(x)g(x) = LN = \lim\limits_{x\to x_o} f(x) \lim\limits_{x\to x_o} g(x)$. ☐

Remark 1.4.4

(a.) Given that $f_1, f_2, \cdots, f_n$ are $n-$real valued-functions of a real variable, and

$$\lim_{x\to x_o} f_1(x),\ \lim_{x\to x_o} f_2(x), \cdots, \lim_{x\to x_o} f_n(x) \quad \text{exist,}$$

then

$$\lim_{x\to x_o} f_1(x)f_2(x)\cdots f_n(x) = \lim_{x\to x_o} f_1(x) \lim_{x\to x_o} f_2(x) \cdots \lim_{x\to x_o} f_n(x). \quad (1.23)$$

That is, *limit of product is product of limits.*

(b.) Following equation (1.23), it is clear that for any positive integer n we have

$$\lim_{x\to x_o} (f(x))^n = \left(\lim_{x\to x_o} f(x)\right)^n, \quad n \in \mathbb{N}. \quad (1.24)$$

Theorem 1.4.5. *Given that* $\lim\limits_{x\to x_o} f(x) = L$ *and* $\lim\limits_{x\to x_o} g(x) = N$, *then*

$$\lim_{x\to x_o} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to x_o} f(x)}{\lim\limits_{x\to x_o} g(x)} = \frac{L}{N} \quad \textit{provided} \quad \lim_{x\to x_o} g(x) \neq 0. \quad (1.25)$$

Proof. We give a simple proof of this theorem as a consequence of limit of product of two functions.
Clearly we can write $f(x) = \dfrac{f(x)g(x)}{g(x)}$ with $g(x) \neq 0$.
Recall that $\lim\limits_{x\to x_o} f(x) = L$ and $\lim\limits_{x\to x_o} g(x) = N$, then

$$L = \lim_{x\to x_o} f(x) = \lim_{x\to x_o} \left(\frac{f(x)}{g(x)} g(x)\right) = \lim_{x\to x_o} \left(\frac{f(x)}{g(x)}\right) \times \lim_{x\to x_o} g(x)$$

$$L = \lim_{x\to x_o} \left(\frac{f(x)}{g(x)}\right) \times \lim_{x\to x_o} g(x) = \lim_{x\to x_o} \left(\frac{f(x)}{g(x)}\right) \times N$$

$$\Longrightarrow \frac{L}{N} = \lim_{x\to x_o}\left(\frac{f(x)}{g(x)}\right).$$

Hence,

$$\lim_{x\to x_o}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_o} f(x)}{\lim_{x\to x_o} g(x)} = \frac{L}{N} \quad \text{provided} \quad \lim_{x\to x_o} g(x) \neq 0.$$

□

Summary of Theorems on Limits

Given that $f, g : X \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ are real-valued functions. If $\lim_{x\to x_o} f(x)$ and $\lim_{x\to x_o} g(x)$ exist, then

(a.) $\lim_{x\to x_o}(f(x)+g(x)) = \lim_{x\to x_o} f(x) + \lim_{x\to x_o} g(x)$;

(b.) $\lim_{x\to x_o} f(x)g(x) = \lim_{x\to x_o} f(x) \lim_{x\to x_o} g(x)$; and

(c.) $\lim_{x\to x_o}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to x_o} f(x)}{\lim_{x\to x_o} g(x)}$ provided $\lim_{x\to x_o} g(x) \neq 0$.

These theorems will also be applied when evaluating the left- and right-hand limits of functions.

Example 1.14

Evaluate $\lim_{x\to x_o} f(x)$ given that

(i.) $f(x) = \begin{cases} 2x+5 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3, \end{cases}$ where $x_o = 3$,

(ii.) $f(x) = \left(\frac{2x^2+5x-3}{x^2+3x-2}\right)$, where $x_o = 1$.

Solutions

(i.) Here we will use theorems (1.4.1)-(1.4.3) as follows:
Firstly, in evaluating $\lim_{x\to 3} f(x)$, we keep in mind that if $x \neq 3$ then $f(x) \neq 6$ but rather $f(x) = 2x+5$ i.e. we shall evaluate $\lim_{x\to 3}(2x+5)$.
Secondly, note that the statement x *approaches* x_o means $\lim_{x\to x_o} x = x_o$

eg. $\lim_{x\to 3} x = 3$.
Hence,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x+5) = \lim_{x\to 3} 2x + \lim_{x\to 3} 5$$
$$= 2\lim_{x\to 3} x + \lim_{x\to 3} 5 = 2\times 3 + 5 = 11.$$

Note: From this example, $\lim_{x\to 3}(2x+5)$ is evaluated by *direct substitution as if* $x = 3$. However, we keep in mind that $x \neq 3$ but sufficiently close to 3.

(ii.) By theorem (1.4.5) and direct substitution we have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(\frac{2x^2+5x-3}{x^2+3x-2}\right) = \frac{\lim_{x\to 1}(2x^2+5x-3)}{\lim_{x\to 1}(x^2+3x-2)}$$
$$= \frac{2(1)^2+5(1)-3}{(1)^2+3(1)-2} = 2.$$

Example 1.15

Given that g is a non-negative real valued function and $\lim_{x\to x_o} g(x) \geq 0$. Show that

$$\lim_{x\to x_o}\sqrt{g(x)} = \sqrt{\lim_{x\to x_o} g(x)}.$$

Hence, evaluate $\lim_{x\to 1}(\sqrt{2x+14} - 4x + 5)$.

Solutions

Clearly $g(x) = \sqrt{g(x)} \times \sqrt{g(x)}$. Then by theorem (1.4.4) we have

$$\lim_{x\to x_o} g(x) = \lim_{x\to x_o}(\sqrt{g(x)}\sqrt{g(x)}) = \lim_{x\to x_o}\sqrt{g(x)} \times \lim_{x\to x_o}\sqrt{g(x)}$$
$$\lim_{x\to x_o} g(x) = \left(\lim_{x\to x_o}\sqrt{g(x)}\right)^2$$
$$\implies \sqrt{\lim_{x\to x_o} g(x)} = \lim_{x\to x_o}\sqrt{g(x)}.$$

By theorem (1.4.3) and direct substitution we have

$$\lim_{x\to 1}(\sqrt{2x+14} - 4x + 5) = \lim_{x\to 1}\sqrt{2x+14} - \lim_{x\to 1} 4x + \lim_{x\to 1} 5$$
$$= \sqrt{\lim_{x\to 1}(2x+14)} - 4 + 5$$
$$= \sqrt{16} - 4 + 5 = 5.$$

Control Problem 1.4.4

1. Suppose $\lim\limits_{x\to x_o} g(x)$ exists. Show that

$$\lim_{x\to x_o}\left(\frac{1}{g(x)}\right)^n = \frac{1}{\left(\lim\limits_{x\to x_o} g(x)\right)^n} \quad \text{for all} \quad n\in\mathbb{N}.$$

2. Suppose the real-valued function f is defined by

$$f(x)=\begin{cases} 2x^3-4x^2+5x-1 & \text{if } x\le 1 \\ 3x^3-x^2+x-4 & \text{if } x>1. \end{cases}$$

Evaluate $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 1^+} f(x)$. Show that these values are 2 and -1 respectively.

1.4.5 Limits of Trigonometric Functions

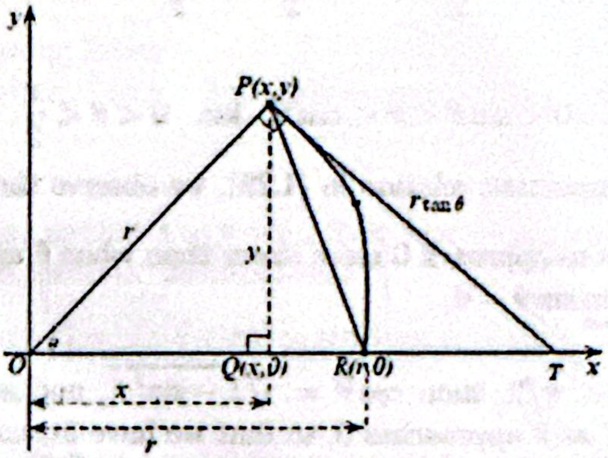

Figure 1.5: Deriving limits of some Trigonometric Functions

In figure 1.5 it is observed that $|OQ| = x$ and $|PQ| = y$ where $x > 0$ and $y > 0$. If $|OP| = r$ then $r^2 = x^2 + y^2$.
It is clear from figure 1.5 that

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad \text{for} \quad 0<\theta<\frac{\pi}{2}. \tag{1.26}$$

In this interval, $\sin\theta, \cos\theta$ and $\tan\theta$ are positive.
From figure 1.5, the areas A_S and A_T of sector and triangle OPR are

$$A_S = \frac{1}{2}r^2\theta > 0 \quad \text{and} \quad A_T = \frac{1}{2}r^2\sin\theta > 0 \tag{1.27}$$

respectively.
Also, it is observed that

$$\tan\theta = \frac{|PT|}{|OP|} = \frac{|PT|}{r} \implies |PT| = r\tan\theta.$$

The area A^* of triangle OPT is

$$A^* = \frac{1}{2}|OP| \times |PT| = \frac{1}{2}r \times r\tan\theta = \frac{1}{2}r^2\tan\theta. \tag{1.28}$$

From figure 1.5, it is clear that area of traingle OPR is less than area of sector OPR which is less than area of triangle OPT i.e.

$$A_T < A_S < A^*$$

which is the same as writing

$$0 < \frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta.$$

This implies

$$0 < \sin\theta < \theta < \tan\theta \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}. \tag{1.29}$$

Following this important relation in (1.29), we observe the following:

(a.) $\sin\theta$ seems to approach 0 more closer than when θ approaches 0. Thus we have $\lim_{\theta\to 0}\sin\theta = 0$,

(b.) If $0 < \theta < \pi/2$, then $\cos\theta = \sqrt{1-\sin^2\theta}$, and so $\sqrt{1-\sin^2\theta}$ approaches 1 as θ approaches 0, so that we have $\lim_{\theta\to 0}\cos\theta = 1$, and

(c.) Noting that $\tan\theta = \sin\theta/\cos\theta$, then by theorem (1.4.5) on limits, we have

$$\lim_{\theta\to 0}\tan\theta = \frac{\lim_{\theta\to 0}\sin\theta}{\lim_{\theta\to 0}\cos\theta} = 0.$$

Hence, we have the following important results:

$$\lim_{x\to 0}\sin x = 0, \quad \lim_{x\to 0}\cos x = 1 \quad \text{and} \quad \lim_{x\to 0}\tan x = 0. \tag{1.30}$$

Divide the inequality (1.29) through by $\sin\theta$, then we have

$$0 < 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \implies \cos\theta < \frac{\sin\theta}{\theta} < 1. \tag{1.31}$$

From this inequality, it is observed that as θ approaches 0, we have $\frac{\sin\theta}{\theta}$ approaches 1 since $\cos\theta$ approaches 1 as θ approaches 0 for $0 < \theta < \pi/2$. Hence, we have

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

Also by theorem (1.4.4),

$$\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta\cos\theta}\right) = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) \times \lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right) = 1.$$

Hence,

$$\lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right) = 1.$$

We now have the following special limits:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1 \quad \text{and} \quad \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1. \tag{1.32}$$

Example 1.16

Establish the following limits
(i.) $\lim_{x\to x_o} \sin x = \sin x_o$ (ii.) $\lim_{x\to x_o} \cos x = \cos x_o$ (iii.) $\lim_{x\to x_o} \tan x = \tan x_o$.

Solutions

(i.) Let $\theta = x - x_o$ so that $x = \theta + x_o$, then as $x \to x_o$, we have $\theta \to 0$. By trigonometric formula, we recall that

$$\sin(x) = \sin(\theta + x_o) = \sin\theta\cos x_o + \sin x_o\cos\theta.$$

Thus, we have

$$\lim_{x\to x_o} \sin x = \lim_{\theta\to 0}\sin(\theta + x_o) = \cos x_o \lim_{\theta\to 0}\sin\theta + \sin x_o \lim_{\theta\to 0}\cos\theta = \sin x_o.$$

Hence, with direct substitution we have $\lim_{x\to x_o} \sin x = \sin x_o$.

(ii.) Recall that $\cos x = \cos(\theta + x_o) = \cos x_o\cos\theta - \sin x_o\sin\theta$ so that

$$\lim_{x\to x_o} \cos x = \lim_{\theta\to 0}\cos(\theta + x_o) = \cos x_o \lim_{\theta\to 0}\cos\theta - \sin x_o \lim_{\theta\to 0}\sin\theta = \cos x_o.$$

Hence by direct substitution, we have $\lim_{x\to x_o} \cos x = \cos x_o$.

(iii.)

$$\lim_{x\to x_o} \tan x = \lim_{x\to x_o}\left(\frac{\sin x}{\cos x}\right) = \frac{\lim_{x\to x_o} \sin x}{\lim_{x\to x_o} \cos x} = \frac{\sin x_o}{\cos x_o} = \tan x_o.$$

Example 1.17

Show that
(i.) $\lim_{x\to 0}\left(\frac{x}{\sin x}\right) = 1$ (ii.) $\lim_{x\to 0}\left(\frac{\tan kx}{x}\right) = k$, where k is a non-zero constant.

Solutions

(i.)

$$\lim_{x\to 0}\left(\frac{x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{1}{(\sin x)/x}\right) = \frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} = \frac{1}{1} = 1.$$

(ii.)

$$\lim_{x\to 0}\left(\frac{\tan kx}{x}\right) = k \times \lim_{x\to 0}\left(\frac{\tan kx}{kx}\right).$$

Let $y = kx$, then it is clear that $y \to 0$ as x approaches 0 so that

$$\lim_{x\to 0}\left(\frac{\tan kx}{x}\right) = k \times \lim_{x\to 0}\left(\frac{\tan kx}{kx}\right) = k \times \lim_{y\to 0}\left(\frac{\tan y}{y}\right) = k \times 1 = k.$$

Control Problem 1.4.5

By evaluating the following limits, verify that
(i.) $\lim_{x\to 0}\left(\frac{\sin 2x}{5x}\right) = \frac{2}{5}$ (ii.) $\lim_{x\to 0}\left(\frac{\sin 7x}{\sin 3x}\right) = \frac{7}{3}$
(iii.) $\lim_{x\to 0}\left(\frac{\tan kx}{\tan mx}\right) = \frac{k}{m}$ k, m constants.

1.4.6 Limit at Infinity

Let x be a real variable. If $x > 0$ *grows* so large without bound, then we say x approaches *plus infinity* and write $x \to +\infty$ or simply $x \to \infty$. If $x < 0$ and the *magnitude* of x *grows* so large without bound then we say x approaches *minus infinity* and write $x \to -\infty$.

The limits L_p and L_m of a function f at plus and minus infinity are written as

$$L_p = \lim_{x\to\infty} f(x) \quad \text{and} \quad L_m = \lim_{x\to -\infty} f(x) \tag{1.33}$$

respectively.

Definition 1.4.5. *Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function.*

(a.) *The real number L_p is called limit of f at plus infinity if given any positive number ϵ, we can find a positive number N depending on ϵ such that*

$$|f(x) - L_p| < \epsilon \quad \textit{for all } x \textit{ satisfying } x > N. \tag{1.34}$$

(b.) *The real number L_m is called limit of f at minus infinity if given any positive number ϵ, we can find a positive number N depending on ϵ such that*

$$|f(x) - L_m| < \epsilon \quad \textit{for all } x \textit{ satisfying } x < -N. \tag{1.35}$$

Example 1.18

Prove that

(i.) $\lim_{x\to\infty}\left(\frac{1}{1+x^2}\right) = 0$ (ii.) $\lim_{x\to-\infty}\left(\frac{x}{1+x}\right) = 1.$

Solutions

(i.) Clearly

$$x > N \implies 1 + N^2 < 1 + x^2 \implies \frac{1}{1+x^2} < \frac{1}{1+N^2}.$$

Then

$$\left|\frac{1}{1+x^2} - 0\right| < \frac{1}{1+N^2}.$$

Choose $\epsilon = \frac{1}{1+N^2}$, then for positive N we have $N = \sqrt{\frac{1-\epsilon}{\epsilon}}$. Clearly for any choice of $\epsilon = \epsilon_1$ with $0 < \epsilon_1 < 1$ we find a positive number $N = N_1 = \sqrt{\frac{1-\epsilon_1}{\epsilon_1}}$, and we have

$$\left|\frac{1}{1+x^2} - 0\right| < \epsilon_1 \quad \text{for all } x > N_1$$

and if $\epsilon = \epsilon_2 > 1$, we can still find positve $N = N_1 = \sqrt{\frac{1-\epsilon_1}{\epsilon_1}}$ with $0 < \epsilon_1 < 1$ such that

$$\left|\frac{1}{1+x^2} - 0\right| < \epsilon_1 < \epsilon_2 = \epsilon \quad \text{for all } x > N_1.$$

Hence, $\lim_{x\to\infty}\left(\frac{1}{1+x^2}\right) = 0.$

(ii.) Note that

$$\left|\frac{x}{x+1}-1\right| = \left|\frac{x-x-1}{x+1}\right| = \frac{1}{|x+1|}.$$

We recall that $|x+1| = -(x+1)$ whenever $x+1<0$. Thus, if $x<-N$ then $-x>N$ which implies $-x-1>N-1$

$$\Longrightarrow \left|\frac{x}{x+1}-1\right| = \frac{1}{|x+1|} = \frac{1}{-x-1} < \frac{1}{N-1}.$$

Choose $\epsilon = \dfrac{1}{N-1} \Longrightarrow N = \dfrac{1}{\epsilon}+1$. Clearly for any chosen $\epsilon>0$ we can find a positive number N such that

$$\left|\frac{x}{x+1}-1\right| < \epsilon \quad \text{for all } x<-N.$$

Hence, $\displaystyle\lim_{x\to-\infty}\left(\frac{x}{1+x}\right) = 1.$

Control Problem 1.4.6

Use definition (1.4.5) to prove that

(i.) $\displaystyle\lim_{x\to-\infty}\frac{1}{x^2} = 0$ (ii.) $\displaystyle\lim_{x\to\infty}\left(\frac{2x+3}{3x-2}\right) = \frac{2}{3}.$

1.4.7 Infinite Limits

Suppose the values $f(x)$ of the real-valued function f *increase* in *magnitude* without bound as real variable x approaches a real number x_o, then we say that the limit of $f(x)$ as $x\to x_o$ is *infinite*. If $f(x)>0$ and limit of f as $x\to x_o$ is infinite, then we write

$$\lim_{x\to x_o} f(x) = +\infty.$$

This is also written as $f(x)\to+\infty$ as $x\to x_o$. Also, if $f(x)<0$ and the limit of f as $x\to x_o$ is infinite, then we write

$$\lim_{x\to x_o} f(x) = -\infty.$$

This is also written as $f(x)\to-\infty$ as $x\to x_o$.

For example, the function f defined by $f(x)=1/x^2$ increases without bound as $x\to0$. This behaviour is seen in figure 1.6, where near $x_o=0$, the graph of $f(x)=1/x^2$ forms *asymptotes* to the vertical axis. The use of the word *asymptote to the vertical axis* means that the curve $y = 1/x^2$ would seem parallel and close to a line parallel to the vertical axis as $x \to 0$. Thus, from figure 1.6 we have

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty.$$

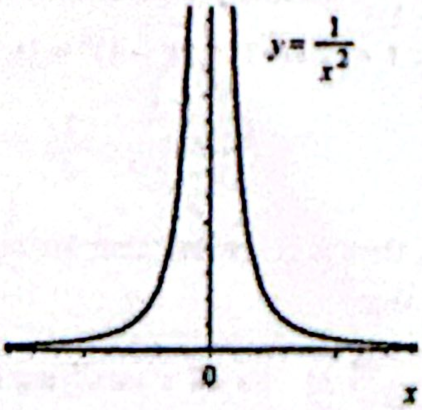

Figure 1.6: Limit of $y = \frac{1}{x^2}$ as $x \to 0$ is an infinite limit.

Definition 1.4.6. *Let $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function. f is said to increase in magnitude without bound as x approaches a real number x_o, if $|f(x)| \neq 0$ and given any positive number M, we can find a positive number $\delta > 0$ such that*

$$|f(x)| > M \quad \text{for all } x \text{ satisfying } 0 < |x - x_o| < \delta. \tag{1.36}$$

(a.) If $f(x) > 0$ as x approaches x_o, then

$$f(x) > M \quad \text{for all } x \text{ satisfying } 0 < |x - x_o| < \delta, \tag{1.37}$$

and we write

$$\lim_{x \to x_o} f(x) = +\infty.$$

(b.) If $f(x) < 0$ as x approaches x_o, then

$$f(x) < -M \quad \text{for all } x \text{ satisfying } 0 < |x - x_o| < \delta, \tag{1.38}$$

and we write

$$\lim_{x \to x_o} f(x) = -\infty.$$

Example 1.19

Show that $\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$.

Solution

Let $f(x) = \frac{1}{(x-1)^2}$. Clearly $f(x) > 0$ for all $x \neq 1$. If $0 < |x-1| < \delta$, then for $0 < \delta < 1$ we have $0 < (x-1)^2 = |x-1|^2 < \delta^2 < \delta$. Clearly we have

$$\frac{1}{(x-1)^2} > \frac{1}{\delta}.$$

Choose $M = \frac{1}{\delta}$, then it is obvious that for any $M > 0$, we can always find $\delta > 0$ such that

$$\frac{1}{(x-1)^2} > M \quad \text{for all } x \text{ satisfying } 0 < |x-1| < \delta.$$

Hence, $\lim_{x\to 1} \frac{1}{(x-1)^2} = +\infty$.

Example 1.20

Evaluate $\lim_{x\to \frac{\pi}{2}^-} \tan x$ and $\lim_{x\to \frac{\pi}{2}^+} \tan x$.

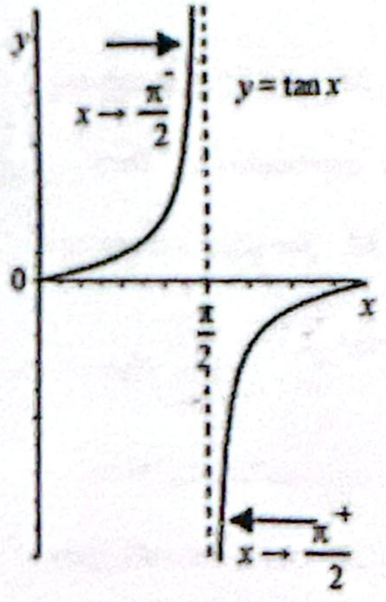

Figure 1.7: limits of $y = \tan x$ as $x \to \pi^+/2$ and $x \to \pi^-/2$.

Solutions

From the graph of $y = \tan x$ (see figure 1.7), it is observed that $\tan x$ approaches $-\infty$ as $x \to \pi^+/2$, while $f(x)$ approaches $+\infty$ as $x \to \pi^-/2$. Hence,

$$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty.$$

Control Problem 1.4.7

Use the graph of the curve $y = \dfrac{1}{x}$ to show that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

1.4.8 Limits of Exponential and Logarithmic Functions

Exponential Function

A real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = b^x \tag{1.39}$$

is called an *exponential function*, where $b > 0$. The graphs of exponential functions are shown for $0 < b < 1$ and $b > 1$ in figure 1.8. Following these graphs, it can be observed that

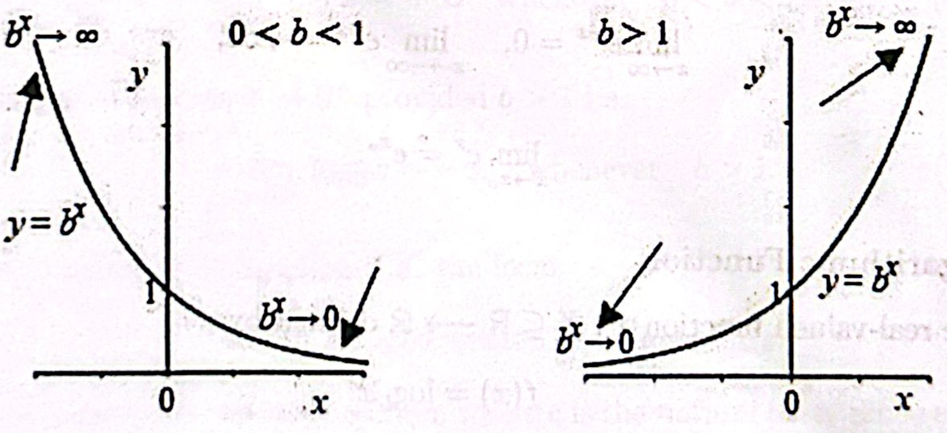

Figure 1.8: (a) $y = b^x$ for $0 < b < 1$. (b) $y = b^x$ for $b > 1$.

(a.) $b^x \to 0$ as $x \to \infty$ provided $0 < b < 1$ i.e.

$$\lim_{x \to \infty} b^x = 0 \quad \text{whenever} \quad 0 < b < 1. \tag{1.40}$$

(b.) $b^x \to \infty$ as $x \to \infty$ provided $b > 1$ i.e.

$$\lim_{x\to\infty} b^x = +\infty \quad \text{whenever} \quad b > 1. \tag{1.41}$$

(c.) $b^x \to \infty$ as $x \to -\infty$ provided $0 < b < 1$ i.e.

$$\lim_{x\to-\infty} b^x = \infty \quad \text{whenever} \quad 0 < b < 1. \tag{1.42}$$

(d.) $b^x \to 0$ as $x \to -\infty$ provided $b > 1$ i.e.

$$\lim_{x\to-\infty} b^x = 0 \quad \text{whenever} \quad b > 1. \tag{1.43}$$

Also, it can be deduced from the graph of an exponential function that given any real number x_o, we have

$$\lim_{x\to x_o} b^x = b^{x_o}. \tag{1.44}$$

A special exponential function f, of the form

$$f(x) = e^x \quad \text{also written as} \quad f(x) = \exp(x) \tag{1.45}$$

is called exponential function with *natural base e*. The natural base e is an irrational number given as $e = 2.71828\cdots$. Observe that $e > 1$, so that by properties of exponential function we have

$$\lim_{x\to\infty} e^x = +\infty, \quad \lim_{x\to-\infty} e^x = 0, \tag{1.46}$$

$$\lim_{x\to\infty} e^{-x} = 0, \quad \lim_{x\to-\infty} e^{-x} = +\infty, \tag{1.47}$$

and

$$\lim_{x\to x_o} e^x = e^{x_o}. \tag{1.48}$$

Logarithmic Function

The real-valued function $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \log_b x \tag{1.49}$$

is called *logarithmic function*, where $x > 0$ and $b > 0$. The logarithmic function is an *inverse* function of the exponential function in the sense

$$b^{\log_b x} = x \quad \text{and} \quad \log_b b^x = x \quad (x > 0, b > 0). \tag{1.50}$$

Following the graphs of logarithmic function in figure 1.9, it can be observed that

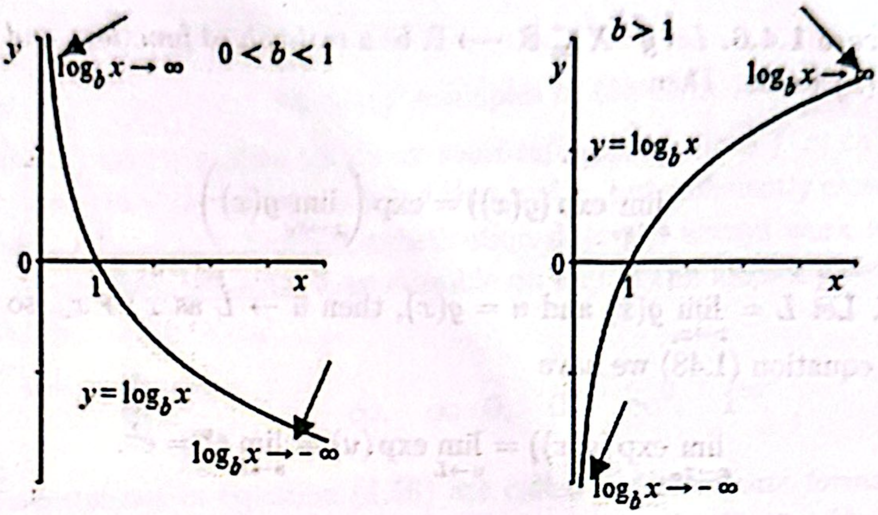

Figure 1.9: (a) $y = \log_b x$ for $0 < b < 1$. (b) $y = \log_b x$ for $b > 1$.

(a.) $\log_b x \to -\infty$ as $x \to \infty$ provided $0 < b < 1$ i.e.

$$\lim_{x\to\infty} \log_b x = -\infty \quad \text{whenever} \quad 0 < b < 1. \tag{1.51}$$

(b.) $\log_b x \to +\infty$ as $x \to \infty$ provided $b > 1$ i.e.

$$\lim_{x\to\infty} \log_b x = +\infty \quad \text{whenever} \quad b > 1. \tag{1.52}$$

(c.) $\log_b x \to +\infty$ as $x \to 0^+$ provided $0 < b < 1$ i.e.

$$\lim_{x\to 0^+} \log_b x = +\infty \quad \text{whenever} \quad 0 < b < 1. \tag{1.53}$$

(d.) $\log_b x \to -\infty$ as $x \to 0^+$ provided $b > 1$ i.e.

$$\lim_{x\to 0^+} \log_b x = -\infty \quad \text{whenever} \quad b > 1. \tag{1.54}$$

A special logarithmic function f of the form

$$f(x) = \log_e x \quad \text{also written as} \quad f(x) = \ln x \tag{1.55}$$

is called *natural logarithmic function*, where e is the *natural base*. Since $e > 1$, then it is clear that

$$\lim_{x\to\infty} \ln x = +\infty, \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty. \tag{1.56}$$

Next, we state and prove the following important theorems concerning the exponential and logarithmic functions with natural base:

Theorem 1.4.6. *Let $g : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function, and suppose $\lim_{x\to x_o} g(x)$ exists. Then*

$$\lim_{x\to x_o} \exp\left(g(x)\right) = \exp\left(\lim_{x\to x_o} g(x)\right). \tag{1.57}$$

Proof. Let $L = \lim_{x\to x_o} g(x)$ and $u = g(x)$, then $u \to L$ as $x \to x_o$, so that by using equation (1.48) we have

$$\lim_{x\to x_o} \exp\left(g(x)\right) = \lim_{u\to L} \exp\left(u\right) = \lim_{u\to L} e^u = e^L.$$

But, $L = \lim_{x\to x_o} g(x)$. Hence, we have

$$\lim_{x\to x_o} \exp\left(g(x)\right) = e^L = \exp\left(\lim_{x\to x_o} g(x)\right).$$

□

Theorem 1.4.7. *Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a positive real-valued function, and suppose $\lim_{x\to x_o} f(x)$ exists and is positive. Then*

$$\lim_{x\to x_o} \ln f(x) = \ln\left(\lim_{x\to x_o} f(x)\right). \tag{1.58}$$

Proof. Recall that

$$\exp\left(\ln f(x)\right) = f(x).$$

Also, by theorem (1.4.6), it is clear that

$$\lim_{x\to x_o} f(x) = \lim_{x\to x_o} \exp(\ln f(x)) = \exp\left(\lim_{x\to x_o} \ln f(x)\right)$$

$$\Longrightarrow \lim_{x\to x_o} f(x) = \exp\left(\lim_{x\to x_o} \ln f(x)\right).$$

Taking natural logarithm on both sides of equation, we have

$$\ln\left(\lim_{x\to x_o} f(x)\right) = \lim_{x\to x_o} \ln f(x).$$

□

1.4.9 Indeterminate Forms

We have seen that there are many examples of the form $L = \lim_{x \to x_o} f(x)$ in which evaluations are done by *direct substitutions* of x_o into $f(x)$ to obtain the value L, although we keep in mind that $x \neq x_o$ but sufficiently close to x_o. However, this method of direct substitution does not always work *initially* for many problems since we may stumble on any of the following seven (7) expressions:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad \infty \cdot 0, \quad 0^0, \quad \infty^0, \quad 1^\infty. \tag{1.59}$$

These expressions in equation (1.59) are called *indeterminate forms* which are obtained when direct substitution is used in evaluating limits of functions.

Example 1.21

Identify through direct substitution the indeterminate forms of the following limits:

(i.) $\lim_{x \to 0} \left(\frac{1 - \cos x}{x} \right)$ (ii.) $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2} \right) \tan x$ (iii.) $\lim_{x \to \infty} \left(\frac{e^x}{x} \right)$

(iv.) $\lim_{x \to 0^+} (1 + x)^{1/x}$.

(i.) Clearly, as $x \to 0$, we have $\cos x$ approaches 1 so that $1 - \cos x$ approaches 0. Hence by direct substitution,

$$\lim_{x \to 0} \left(\frac{1 - \cos x}{x} \right) \text{ has the indeterminate form } \frac{0}{0}.$$

(ii.) Clearly as $x \to \pi/2$, we have $x - \pi/2$ approaches 0, and $\tan x \to \pm\infty$. Hence, by direct substitution

$$\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2} \right) \tan x \text{ has the indeterminate form } 0 \cdot \infty.$$

(iii.) Clearly $e^x \to \infty$ as $x \to \infty$ so that

$$\lim_{x \to \infty} \left(\frac{e^x}{x} \right) \text{ has the indeterminate form } \frac{\infty}{\infty}.$$

(iv.) Clearly, as $x \to 0^+$, we have $1 + x$ approaches 1, and $1/x \to +\infty$. Hence,

$$\lim_{x \to 0^+} (1 + x)^{1/x} \text{ has the indeterminate form } 1^\infty.$$

Example 1.21 above demonstrates how indeterminate forms are obtained using direct substitution. An indeterminate form obtained upon evaluation of a limit implies that further simplifications should be made to know the value of the limit which is not of the indeterminate forms.

1.4.10 Evaluation of Limits

This section considers various examples on how to evaluate limits of functions using theorems on limits. We shall also demonstrate how to evaluate limits with indeterminate forms.

Example 1.22

Evaluate the following limits
(i.) $\lim\limits_{x\to 3} 1^{1/(x-3)}$ (ii.) $\lim\limits_{x\to 3^-} 3^{1/(x-3)}$.

Solutions

(i.) Since $x \neq 3$ then $\dfrac{1}{x-3}$ is a real number. This implies that $1^{1/(x-3)} = 1$ so that we have

$$\lim_{x\to 3} 1^{1/(x-3)} = \lim_{x\to 3} 1 = 1.$$

(ii.) Since $x \to 3^-$, then $x < 3 \implies x - 3 < 0$ so that $\dfrac{1}{x-3} < 0$. Hence, as $x \to 3^-$, we have $\dfrac{1}{x-3} \to -\infty$. This means that as $x \to 3^-$, we have $3^{1/(x-3)}$ becomes very small i.e. $3^{1/x-3} \to 0$ as $\dfrac{1}{x-3} \to -\infty$ as $x \to 3^-$.
Hence, $\lim\limits_{x\to 3^-} 3^{1/x-3} = 0$.

Example 1.23

Evaluate the following limits
(i.) $\lim\limits_{x\to -2} \sqrt[3]{2x^3 - 11}$ (ii.) $\lim\limits_{x\to 2} \left(\dfrac{x^2 - x + 3}{x^3 - x^2 + 3}\right)$.

Solutions

(i.) By direct substitution, we have

$$\lim_{x\to -2} \sqrt[3]{2x^3 - 11} = \sqrt[3]{-2\times 8 - 11} = \sqrt[3]{-27} = -3.$$

(ii.) By direct substitution, we have

$$\lim_{x\to 2}\left(\frac{x^2-x+3}{x^3-x^2+3}\right)=\frac{2^2-2+3}{2^3-2^2+3}=\frac{5}{7}.$$

Example 1.24

Evaluate the following limits

(i.) $\lim_{x\to 1} f(x)$ given that $f(x)=\begin{cases}\dfrac{x^2+x-2}{x^2-x} & \text{if } x\neq 1\\ 2 & \text{if } x=1,\end{cases}$

(ii.) $\lim_{x\to 3}\left(\dfrac{\sqrt{x}-\sqrt{3}}{x-3}\right).$

Solutions

(i.) Given that $f(x)=\begin{cases}\dfrac{x^2+x-2}{x^2-x} & \text{if } x\neq 1\\ 2 & \text{if } x=1,\end{cases}$ in evaluating $\lim_{x\to 1} f(x)$, we note that $x\neq 1$ so that

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\left(\frac{x^2+x-2}{x^2-x}\right).$$

By direct substitution,

$$\lim_{x\to 1}\left(\frac{x^2+x-2}{x^2-x}\right)\quad\text{has indeterminate form}\quad\frac{0}{0}.$$

Factorizing $(x-1)$ from the numerator and denominator through $x^2+x-2=(x-1)(x+2)$ and $x^2-x=x(x-1)$, then we have

$$\begin{aligned}\lim_{x\to 1}\left(\frac{x^2+x-2}{x^2-x}\right)&=\lim_{x\to 1}\left(\frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}}\right)\\&=\lim_{x\to 1}\left(\frac{x+2}{x}\right)=\frac{1+2}{1}=3.\end{aligned}$$

(ii.) By direct substitution $\lim_{x\to 3}\left(\dfrac{\sqrt{x}-\sqrt{3}}{x-3}\right)$ has indeterminate form $\dfrac{0}{0}$.

By rationalizing, we have

$$\begin{aligned}\lim_{x\to 3}\left(\frac{\sqrt{x}-\sqrt{3}}{x-3}\right)&=\lim_{x\to 3}\left(\frac{(\sqrt{x}-\sqrt{3})}{(x-3)}\times\frac{(\sqrt{x}+\sqrt{3})}{(\sqrt{x}+\sqrt{3})}\right)\\&=\lim_{x\to 3}\left(\frac{1}{\sqrt{x}+\sqrt{3}}\right)=\frac{1}{\sqrt{3}+\sqrt{3}}=\frac{1}{2\sqrt{3}}.\end{aligned}$$

Example 1.25

Evaluate the following limits

(i.) $\lim_{x\to 0}\left(\frac{\csc x-\cot x}{x}\right)$, where $\csc x=\frac{1}{\sin x}$ and $\cot x=\frac{1}{\tan x}$

(ii.) $\lim_{x\to \frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)\tan x$.

Solutions

(i.) By direct substitution, $\lim_{x\to 0}\left(\frac{\csc x-\cot x}{x}\right)$ has the indeterminate form $\frac{\infty-\infty}{0}$. By further simplification we have

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\csc x-\cot x}{x}\right) &= \lim_{x\to 0}\left(\frac{\frac{1}{\sin x}-\frac{\cos x}{\sin x}}{x}\right)=\lim_{x\to 0}\left(\frac{1-\cos x}{x\sin x}\right)\\
&= \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{2x\sin\frac{x}{2}\cos\frac{x}{2}}\right)=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x\cos\frac{x}{2}}\right)\\
&= \lim_{x\to 0}\left(\frac{\tan\frac{x}{2}}{x}\right)=\lim_{x\to 0}\left(\frac{\tan\frac{x}{2}}{\frac{1}{2}x}\times\frac{1}{2}\right)\\
&= \frac{1}{2}\left[\lim_{x\to 0}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\right].
\end{aligned}$$

If $y=\frac{x}{2}$, then $y\to 0$ as $x\to 0$, so that

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\csc x-\cot x}{x}\right) &= \frac{1}{2}\left[\lim_{x\to 0}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\right]\\
&= \frac{1}{2}\left[\lim_{y\to 0}\left(\frac{\tan y}{y}\right)\right]=\frac{1}{2}\times 1=\frac{1}{2}.
\end{aligned}$$

(ii.) By direct substitution $\lim_{x\to\frac{\pi}{2}}\left(x-\frac{\pi}{2}\right)\tan x$ has indeterminate form $0\cdot\infty$. We recall that

$$\tan x=\frac{\sin x}{\cos x}=\frac{\cos\left(x-\frac{\pi}{2}\right)}{-\sin\left(x-\frac{\pi}{2}\right)}.$$

Let $y = x - \frac{\pi}{2}$, then $y \to 0$ as $x \to \frac{\pi}{2}$. Clearly,

$$\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x = \lim_{y \to 0} \left(\frac{y \cos y}{-\sin y}\right) = -\lim_{y \to 0} \left(\frac{y}{\sin y}\right) \times \lim_{y \to 0} \cos y$$
$$= -1 \times 1 = -1.$$

Example 1.26

Evaluate the following limits

(i.) $\lim_{x \to -\infty} \sqrt{\frac{x^2+1}{4x^2+3x-2}}$ (ii.) $\lim_{x \to \infty} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$.

Solutions

(i.) By direct substitution $\lim_{x \to -\infty} \sqrt{\frac{x^2+1}{4x^2+3x-2}}$ has the indeterminate form $\frac{\infty}{\infty}$. We recall that $\frac{1}{x} \to 0$ and $\frac{1}{x^2} \to 0$ as $x \to -\infty$. Clearly,

$$\lim_{x \to -\infty} \sqrt{\frac{x^2+1}{4x^2+3x-2}} = \sqrt{\lim_{x \to -\infty} \left(\frac{x^2+1}{4x^2+3x-2}\right)}$$
$$= \sqrt{\lim_{x \to -\infty} \left(\frac{1 + \frac{1}{x^2}}{4 + \frac{3}{x} - \frac{2}{x^2}}\right)}$$
$$= \sqrt{\left(\frac{1+0}{4+0-0}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

(ii.) By direct substitution, $\lim_{x \to \infty} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$ has indeterminate form $\frac{\infty}{\infty}$. We note that $e^{-x} \to 0$ and $e^{-2x} \to 0$ as $x \to \infty$. Clearly,

$$\lim_{x \to \infty} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right) = \lim_{x \to \infty} \left(\frac{\frac{e^x + e^{-x}}{e^x}}{\frac{e^x - e^{-x}}{e^x}}\right) = \lim_{x \to \infty} \left(\frac{1 + e^{-2x}}{1 - e^{-2x}}\right)$$
$$= \frac{1+0}{1-0} = 1.$$

In the examples that will follow, we shall consider limits with indeterminate form 1^∞. A special limit will be helpful:

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to 0}(1+kx)^{1/x} = e^k, \tag{1.60}$$

where k is a constant.

Also, the following theorem will be useful.

Theorem 1.4.8. *Let f be a real-valued function defined by $f(x) = p(x)^{q(x)}$, where $p(x)$ and $q(x)$ are real-valued functions satisfying*

$$\lim_{x\to x_o} p(x) = 1 \quad \textit{and} \quad \lim_{x\to x_o} q(x) = +\infty,$$

so that $\lim_{x\to x_o} f(x)$ has the indeterminate form 1^∞ then

$$\lim_{x\to x_o} p(x)^{q(x)} = \exp\left(\lim_{x\to x_o}(p(x)-1)q(x)\right). \tag{1.61}$$

Proof. Let $p(x) = 1+u(x)$ such that $u(x)\to 0$ as $x\to x_o$. We shall write $u = u(x)$.

$$\begin{aligned}\lim_{x\to x_o} p(x)^{q(x)} &= \lim_{x\to x_o}(1+u)^{q(x)} = \lim_{x\to x_o}\left((1+u)^{1/u}\right)^{uq(x)}\\ &= \left(\lim_{x\to x_o}(1+u(x))^{1/u(x)}\right)^{\lim_{x\to x_o} u(x)q(x)}\\ &= \left(\lim_{u\to 0}(1+u)^{1/u}\right)^{\lim_{x\to x_o} u(x)q(x)}.\end{aligned}$$

By equation (1.60) and the fact that $u(x) = p(x)-1$, we have

$$\lim_{x\to x_o} p(x)^{q(x)} = \left(\lim_{u\to 0}(1+u)^{1/u}\right)^{\lim_{x\to x_o} u(x)q(x)} = \exp\left(\lim_{x\to x_o}(p(x)-1)q(x)\right).$$

□

Example 1.27

Evaluate the following limits

(i.) $\displaystyle\lim_{x\to\infty}\left(\frac{x-1}{x+3}\right)^{x+2}$

(ii.) $\displaystyle\lim_{x\to 0}(\cos x)^{1/x}$.

Solutions

(i.) Let $p(x) = \dfrac{x-1}{x+3}$ and $q(x) = x+2$, then it clear that

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty}\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty}\left(\frac{1-1/x}{1+3/x}\right) = 1$$

and $q(x) = x+2 \to \infty$ as $x \to \infty$.
By theorem (1.4.8), we have

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x-1}{x+3}\right)^{x+2} &= \exp\left(\lim_{x\to\infty}\left((p(x)-1)q(x)\right)\right) \\
&= \exp\left(\lim_{x\to\infty}\left(\left(\frac{x-1}{x+3}-1\right)(x+2)\right)\right) \\
&= \exp\left(\lim_{x\to\infty}\left(\frac{-4(x+2)}{x+3}\right)\right) \\
&= \exp\left(-4\lim_{x\to\infty}\left(\frac{x+2}{x+3}\right)\right) \\
&= \exp(-4\times 1) = e^{-4}.
\end{aligned}$$

(ii.) By direct substitution, $\lim_{x\to 0}(\cos x)^{1/x}$ has indeterminate form 1^∞. By theorem (1.4.8), we have

$$\lim_{x\to 0}(\cos x)^{1/x} = \exp\left(\lim_{x\to 0}(\cos x - 1)\times 1/x\right).$$

We note that

$$\lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right) = \lim_{x\to 0}\left(\frac{-2\sin(x/2)\sin(x/2)}{x}\right) = 0.$$

Thus, we have

$$\lim_{x\to 0}(\cos x)^{1/x} = \exp\left(\lim_{x\to 0}(\cos x - 1)\times 1/x\right) = \exp(0) = e^0 = 1.$$

Examples on indeterminate forms 0^0 and ∞^0 will be considered in chapter 2.

Exercise 1.4

1. By using the $\epsilon-\delta$ definition, prove that

(i.) $\lim_{x\to 1}(x^3+x+1) = 3$ (ii.) $\lim_{x\to 2}(x^2-6x+4) = -4$

(iii.) $\lim_{x\to -1}\left(\dfrac{1}{2x-5}\right) = -\dfrac{1}{7}$ (iv.) $\lim_{x\to 0} x^2 \neq 1$

(v.) $\lim_{x\to 0}(x^2-3x+2) \neq 3$

2. Given that the real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x^2 - 9} & \text{if } x \neq 3 \\ 1 & \text{if } x = 3. \end{cases}$$

Evaluate $\lim_{x \to 3} f(x)$.

3. Evaluate the following:

(i.) $\lim_{x \to 4} \left(\dfrac{x^3 - 64}{x - 4} \right)$ (ii.) $\lim_{x \to 1} \left(\dfrac{2 - x - x^2}{3x^2 + x - 1} \right)$

(iii.) $\lim_{x \to 2} \sqrt[3]{2x^3 - 3x - 18}$.

4. Evaluate without using differentiation the following:

(i.) $\lim_{x \to \infty} \left(\dfrac{3x^2 + 2x + 5}{3 + x - 2x^2} \right)$ (ii.) $\lim_{x \to 9} \left(\dfrac{3 - \sqrt{x}}{x - 9} \right)$

5. Justify why the following limits do not exist:

(i.) $\lim_{x \to 1} (x - 1) \ln(1 - x)$ (ii.) $\lim_{x \to 0} x^x$ (iii.) $\lim_{x \to 1} 2^{1/(x-1)}$.

6. Given that $f(x) = \dfrac{x - |x|}{x}$.

Evaluate

(i.) $\lim_{x \to 0^+} f(x)$ (ii.) $\lim_{x \to 0^-} f(x)$

7. Given that

$$g(x) = \begin{cases} \dfrac{-|x| + 3}{x^2 - 9} & \text{if } x < 3 \\ \operatorname{sgn}(3 - x) \left(\dfrac{x^2 - 9}{\sqrt{x} - \sqrt{3}} \right) & \text{if } x > 3. \end{cases}$$

Evaluate

(i.) $\lim_{x \to 3^-} g(x)$ (ii.) $\lim_{x \to 3^+} g(x)$

8. Evaluate the following limits without taking derivatives:

(i.) $\lim_{x \to 0} \left(\dfrac{\cos x - \cos\left(\frac{x}{2}\right)}{x^2} \right)$ (ii.) $\lim_{x \to 0} \left(\dfrac{1 - 2\cos x + \cos 2x}{x^2} \right)$

(iii.) $\lim_{x \to 3} \left(\dfrac{x - 3}{\sin \pi x} \right)$ (iv.) $\lim_{x \to \infty} x \sin\left(\dfrac{1}{x} \right)$

(v.) $\lim_{x \to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x}$ (vi.) $\lim_{x \to 0} \left(\dfrac{\sqrt[3]{8 + x} - 2}{x} \right)$

(vii.) $\lim\limits_{x\to 0}\left(\dfrac{b^x - a^x}{x}\right)$, where a and b are positive constants.

$\left[\text{Hint: } \lim\limits_{x\to 0}\left(\dfrac{e^{kx}-1}{x}\right) = k, \quad k \text{ being constant.}\right]$

9. Evaluate the following limits

(i.) $\lim\limits_{x\to 0}(1-\sin^2 x)^{1/(2x^2)}$ (ii.) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

(iii.) $\lim\limits_{x\to\infty}\left(1-\dfrac{3}{x}\right)^{2x}$ (iv.) $\lim\limits_{x\to\infty} x\ln\left(\dfrac{x+3}{x-3}\right)$.

1.5 Continuity of Functions

Let $f, g : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be two real-valued functions defined by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \begin{cases} 2x-1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$$

From these definitions, it is clear that $f(1) = 1^2 = 1$ and $g(1) = 2$. Also,

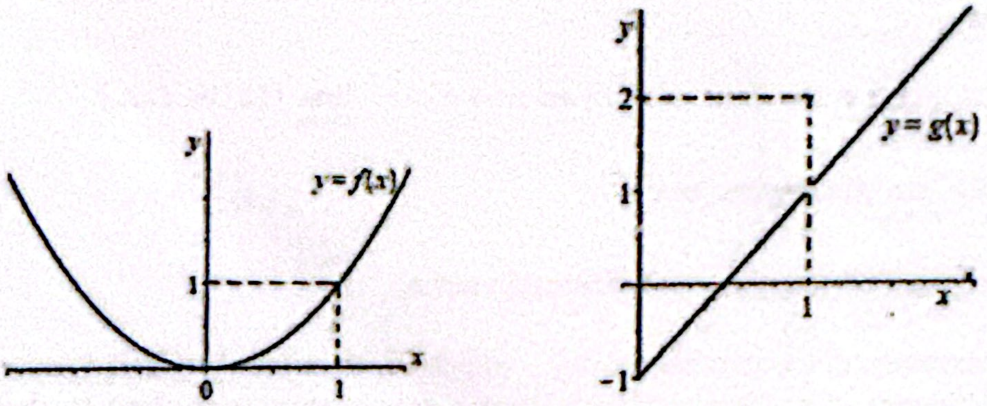

Figure 1.10: (a) $y = f(x) = x^2$. (b) $y = g(x)$.

the limits of $f(x)$ and $g(x)$ as $x \to 1$ are

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 = 1 \quad \text{and} \quad \lim_{x\to 1} g(x) = \lim_{x\to 1}(2x-1) = 1.$$

Figure 1.10 shows the graph of f which can be drawn continuously with a *pencil* by not lifting it at $x = 1$, so that $\lim\limits_{x\to 1} f(x) = f(1)$. Here we say f is *continuous* at $x = 1$. But, the graph of g cannot be drawn continuously without lifting the pencil at $x = 1$ which gives the reason for $\lim\limits_{x\to 1} g(x) \neq g(1)$, and we say g is *discontinuous* at $x = 1$.

To be more precise, if f is a real-valued function defined at $x = x_o$ and $\lim\limits_{x \to x_o} f(x)$ exists, then f is said to be continuous at $x = x_o$ if

$$\lim_{x \to x_o} f(x) = f(x_o). \tag{1.62}$$

1.5.1 Necessary and Sufficient Conditions

Definition 1.5.1. *Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function whose value at $x \in X$ is $f(x)$. f is said to be continuous at $x_o \in X$ if given any positive number $\epsilon > 0$, there exists a positive number $\delta > 0$ depending on ϵ and possibly on x_o such that*

$$|f(x) - f(x_o)| < \epsilon \quad \textit{whenever} \quad |x - x_o| < \delta. \tag{1.63}$$

Remark 1.5.1

Observe from definition (1.5.1), if f is continuous at x_o, then it is *necessary* that

(a.) $f(x_o)$ is the limit of $f(x)$ as $x \to x_o$ i.e. $\lim\limits_{x \to x_o} f(x) = f(x_o)$;

(b.) $\lim\limits_{x \to x_o} f(x)$ exists; and

(c.) $x_o \in X$ i.e. $f(x_o)$ is finite and unique.

Conversely, if the conditions (a.) - (c.) hold, then it is *sufficient* to say, f is continuous at x_o. Hence, the following definition is equivalent to definition (1.5.1):

Definition 1.5.2 (Necessary and Sufficient Conditions for Continuity). *The real-valued function $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_o \in X$ if and only if*

(a.) $f(x_o) \in \mathbb{R}$ and unique i.e. $f(x_o)$ is defined;

(b.) $\lim\limits_{x \to x_o} f(x)$ exists i.e. $\lim\limits_{x \to x_o^-} f(x) = \lim\limits_{x \to x_o^+} f(x)$; and

(c.) $\lim\limits_{x \to x_o} f(x) = f(x_o)$.

Remark 1.5.2

If any of the three conditions (a.) - (c.) in definition (1.5.2) does not hold, then f is said to be *discontinuous* at x_o, and accordingly, x_o is called a *point of discontinuity* of the function f.

Example 1.28

Investigate the continuity of the following real-valued functions f and g at x_o:

(i.) $f(x) = \dfrac{\sin x}{x}, \quad x_o = 0.$

(ii.) $g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2, \end{cases} \qquad x_o = 2.$

Solutions

(i.) Given that $f(x) = \dfrac{\sin x}{x}$, then at $x_o = 0$, it is clear that $f(0)$ is not defined as division by zero is meaningless i.e. $\dfrac{\sin 0}{0}$ is not defined. Hence, condition (a.) in definition (1.5.2) is not satisfied and f is not continuous at $x = 0$ i.e. f is discontinuous at $x = 0$.

(ii.) Given that $g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2, \end{cases}$. It is clear that g is defined at $x_o = 2$ because $g(2) = 4$, so condition (a.) of definition (1.5.2) is satisfied.
Next, investigate the existence of $\lim\limits_{x\to 2} g(x)$. Clearly,

$$\begin{aligned}\lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^-}\left(\frac{(x-2)(x+2)}{x-2}\right) \\ &= \lim_{x\to 2^-}(x+2) = 4\end{aligned}$$

and

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2^+}(x+2) = 4.$$

Thus, we have $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^+} g(x) = 4$ so that $\lim\limits_{x\to 2} g(x)$ exists and is equal to 4. Condition (b.) of definition (1.5.2) is satisfied.

Lastly, we observed that

$$\lim_{x\to 2} g(x) = g(2) = 4.$$

Condition (c.) of definition (1.5.2) is satisfied.
Hence, g is continuous at $x_o = 2$.

Example 1.29

Investigate the continuity of the function f at $x_o = 0$ given that

$$f(x) = \begin{cases} \dfrac{x+|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution

Recall that if $x \neq 0$ then $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0. \end{cases}$ This implies that

$$x + |x| = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \implies \frac{x+|x|}{x} = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

$f(x)$ can be written as

$$f(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly f is defined at $x = 0$ since $f(0) = 0$. Also,

$$\lim_{x\to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 2.$$

The right-hand and left-hand limits are not equal. Thus, $\lim_{x\to 0} f(x)$ does not exist.
Hence, f is not continuous at $x_o = 0$.

Example 1.30

Determine the values of α and β for which the real-valued function f is continuous at $x = 1$ and $x = 2$ given that

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ \alpha x^2 + \beta x + 2 & \text{if } 1 \leq x \leq 2 \\ \alpha - \beta x & \text{if } x > 2. \end{cases}$$

Solution

If f is continuous at $x = 1$ and $x = 2$, then f is defined at $x = 1$ and $x = 2$ with values

$$f(1) = \alpha + \beta + 2 \quad \text{and} \quad f(2) = 4\alpha + 2\beta + 2.$$

If $x < 1$, then $f(x) = 2x$ and $\lim_{x \to 1^-} f(x) = 2$. If f is continuous at $x = 1$, then

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) \implies \alpha + \beta + 2 = 2. \tag{1.64}$$

Also, if $x > 2$, then $f(x) = \alpha - \beta x$ and $\lim_{x \to 2^+} f(x) = \alpha - 2\beta$. If f is continuous at $x = 2$, then

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2) \implies \alpha - 2\beta = 4\alpha + 2\beta + 2. \tag{1.65}$$

Solving the simultaneous equations (1.64) and (1.65), we will obtain $\alpha = 2$ and $\beta = -2$. Hence, f is continuous at $x = 1$ and $x = 2$ if it is defined as

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 2x^2 - 2x + 2 & \text{if } 1 \le x \le 2 \\ 2 + 2x & \text{if } x > 2. \end{cases}$$

Control Problem 1.5.1

Verify that the real-valued function f defined by

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

1.5.2 Continuity Over an Interval

The interval $a \le x \le b$ is called *closed interval*. It includes the end points $a, b \in \mathbb{R}$ with $a < b$. It is commonly denoted as $[a, b]$ and defined as

$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}.$$

The interval $a < x < b$ is called *open interval*. It does not include the end-points a and b. It is commonly denoted as (a, b) and defined as

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Other intervals of interest are denoted as $[a,b)$ and $(a,b]$ defined as

$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\} \quad \text{and} \quad (a,b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

When we refer to an interval, we shall mean any of the forms

$$[a,b], \quad (a,b), \quad [a,b), \quad (a,b].$$

The set $\mathbb{R}$ of real numbers is also the interval $-\infty < x < \infty$.

Definition 1.5.3. *Let $I \subseteq \mathbb{R}$ be an interval. A real-valued function $f : I \longrightarrow \mathbb{R}$ is said to be continuous on I if it is continuous at every point of I.*

If x_o is an arbitrary (any) point of I, then continuity of f on I means

$$\lim_{x \to x_o} f(x) = f(x_o) \quad \text{for every } x_o \in I. \tag{1.66}$$

Let $\Delta x = x - x_o$, then $\Delta x \to 0$ as $x \to x_o$. Also, $x = x_o + \Delta x$ so that $f(x) = f(x_o + \Delta x)$. Equation (1.66) can be written as

$$\lim_{\Delta x \to 0} f(x_o + \Delta x) = f(x_o) \implies \lim_{\Delta x \to 0} (f(x_o + \Delta x) - f(x_o)) = 0.$$

Let $\Delta f = f(x_o + \Delta x) - f(x_o)$, then f is said to be continuous on the interval I if

$$\lim_{\Delta x \to 0} \Delta f = 0 \quad \text{for every } x \in I. \tag{1.67}$$

Hence, to show that f is continuous at every point of an interval I, it suffices to show that

(a.) f is defined at every point of I, and

(b.) $\lim_{\Delta x \to 0} \Delta f = 0$ for every $x \in I$.

If any of these two conditions does not hold, then we say f is *not continuous* in I.

Example 1.31

Investigate the continuity of f defined by $f(x) = x^2$ over $\mathbb{R}$.

Solution

It suffices to show that f is defined for all $x \in \mathbb{R}$ and equation (1.67) holds. No doubt, domain of f is simply $\mathbb{R}$ i.e. $f(x) = x^2$ is defined for every real number x. Also,

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} (f(x + \Delta x) - f(x)) = \lim_{\Delta x \to 0} ((x + \Delta x)^2 - x^2) = 0.$$

Hence, f is continuous for every $x \in \mathbb{R}$.

Control Problem 1.5.2

Show that the real-valued function f defined by

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$

is not continuous in the interval $-1 < x < 1$.

1.5.3 Some Kinds of Discontinuities

Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be real-valued function of a real variable $x \in \mathbb{R}$. Now consider the following kinds of discontinuities:

1. Removable Discontinuity

The function f is said to have *removable discontinuity* at a point $x_o \in \mathbb{R}$ if f is not defined at x_o, but $\lim\limits_{x \to x_o^-} f(x)$ and $\lim\limits_{x \to x_o^+} f(x)$ are finite and are the same i.e. $\lim\limits_{x \to x_o} f(x)$ exists. The use of the word *removable* implies that f can be re-defined so that it becomes continuous at x_o i.e. discontinuity can be removed.

Example 1.32

Show that the function g defined by $g(x) = \dfrac{x^2-1}{x-1}$ has removable discontinuity at $x = 1$. Hence re-define g so that it is continuous at $x = 1$.

Solution

Clearly, at $x = 1$, g is not defined since $g(1)$ has the form $\dfrac{0}{0}$ which is meaningless. Hence, g is not continuous at $x = 1$. But, $\lim\limits_{x \to 1} g(x)$ exists since,

$$\lim_{x \to 1^-} \left(\frac{x^2-1}{x-1} \right) = \lim_{x \to 1^-} (x+1) = 2$$

and

$$\lim_{x \to 1^+} \left(\frac{x^2-1}{x-1} \right) = \lim_{x \to 1^+} (x+1) = 2 \Longrightarrow \lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x).$$

Hence, g has removable discontinuity at $x = 1$. For g to be continuous at $x = 1$, g can be re-defined as a new function g^* given as $g^*(x) = x+1$, where

$$g^*(x) = x + 1 = \frac{(x+1)(x-1)}{x-1} = \frac{x^2-1}{x-1} = g(x) \text{ for } x \neq 1$$

and $g^*(1) = 2 = \lim_{x\to 1} g^*(x)$. Hence, g^* is continuous at $x = 1$.

Example 1.33

Show that the function f defined by $f(x) = x\sin\left(\frac{1}{x}\right)$ has removable discontinuity at $x = 0$. Hence, re-define f so that it is continuous at $x = 0$.

Solution

It is a known fact that direct division of a quantity by zero is meaningless, and thus, if $f(x) = x\sin\left(\frac{1}{x}\right)$, then $f(0)$ is not defined since $\sin\left(\frac{1}{0}\right)$ is not defined. Hence f is not continuous at $x = 0$. Now, let us investigate limit of $f(x)$ as x approaches 0. To do this, we keep in mind that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad \text{for all } x \neq 0.$$

Thus, following this bound for the sine function, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0.$$

Hence, f has removable discontinuity at $x = 0$. The function f can be re-defined as a new function f^* defined as follows

$$f^*(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, if $x \neq 0$, then $f^*(x) = f(x)$, and $f^*(0) = 0$ so that

$$f^*(0) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$

Hence, f^* is continuous at $x = 0$.

2. Jump Discontinuity

The function f is said to have *jump discontinuity* at x_o if $\lim\limits_{x\to x_o^-} f(x)$ and $\lim\limits_{x\to x_o^+} f(x)$ are finite and

$$\lim_{x\to x_o^-} f(x) \neq \lim_{x\to x_o^+} f(x).$$

The *jump* of $f(x)$ at x_o is denoted as $[[f(x_o)]]$ and defined as

$$[[f(x_o)]] = \lim_{x\to x_o^+} f(x) - \lim_{x\to x_o^-} f(x). \tag{1.68}$$

For example, the function $f(x) = \begin{cases} x+1 & \text{if } x > 2 \\ x-2 & \text{if } x \leq 2 \end{cases}$ has jump discontinuity at $x = 2$ because $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$. That is

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-2) = 0$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) = 3.$$

The jump of $f(x)$ at $x = 2$ is simply $[[f(2)]] = 3 - 0 = 3$.

3. Infinite Discontinuity

The function f is said to have infinite discontinuity at $x = x_o$ if either $\lim\limits_{x\to x_o^-} f(x)$ or $\lim\limits_{x\to x_o^+} f(x)$ is infinite or both limits are infinite.

For example the function f defined by $f(x) = \dfrac{1}{x^2}$ has infinite discontinuity at $x = 0$ since

$$\lim_{x\to 0^-}\left(\frac{1}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{1}{x^2}\right) = +\infty.$$

Control Problem 1.5.3

The real-valued function f defined by $f(x) = \ln x$ is not continuous at $x = 0$. Why? Do you think it is possible to define a continuous function which satisfies $f^*(x) = f(x)$ for all $x > 0$ and that f^* is continuous at $x = 0$? Give reasons for your answer.

1.5.4 Some Theorems on Continuity

Theorem 1.5.1. *Polynomial, exponential, sine and cosine functions are continuous on* $\mathbb{R}$.

Proof. The polynomial function f of degree n has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_o,$$

where $a_n \neq 0$. The polynomial function f is defined for all $x \in \mathbb{R}$.
Also,

$$\begin{aligned}\lim_{\Delta x \to 0} \Delta f &= \lim_{\Delta x \to 0} (f(x+\Delta x) - f(x)) \\ &= \lim_{\Delta x \to 0} \Big[a_n\left((x+\Delta x)^n - x^n\right) \\ &\qquad + a_{n-1}\left((x+\Delta x)^{n-1} - x^{n-1}\right) \\ &\qquad\qquad + \cdots + a_1((x+\Delta x) - x)\Big] = 0.\end{aligned}$$

Hence, polynomial function is continuous on $\mathbb{R}$.
The exponential function f defined by $f(x) = b^x$ with $b > 0$ is defined for all $x \in \mathbb{R}$. Clearly,

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} \left(b^{x+\Delta x} - b^x\right) = 0.$$

Hence, the exponential function is continuous on $\mathbb{R}$.
The sine function f is defined by $f(x) = \sin x$. It is defined for all $x \in \mathbb{R}$.
Also,

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} (\sin(x+\Delta x) - \sin x) = 0.$$

Similarly the cosine function $f(x) = \cos x$ is defined for all $x \in \mathbb{R}$ and

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} (\cos(x+\Delta x) - \cos x) = 0.$$

Hence, the sine and cosine functions are continuous on $\mathbb{R}$. ☐

Remark 1.5.3

We keep in mind that the logarithmic function f defined by $f(x) = \log_b x$ is not continuous on $\mathbb{R}$. For instance, $\log_b x$ is not defined at $x = 0$.

Theorem 1.5.2. *Let* $f, g : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous functions at point* $x_o \in X$, *then*

(a.) $f+g$ *defined by* $f(x) + g(x)$

(b.) kf defined by $kf(x)$, k a constant

(c.) fg defined by $f(x)g(x)$

(d.) $\frac{f}{g}$ defined by $\frac{f(x)}{g(x)}$ where $g(x) \neq 0$

are continuous at x_o.

Proof. Here, we shall use theorem on limit of sum to prove (a.) and leave (b.) to (d.) as exercises.

(a.) Since f and g are continuous at x_o, then f and g are defined at x_o and

$$\lim_{x \to x_o} f(x) = f(x_o) \quad \text{and} \quad \lim_{x \to x_o} g(x) = g(x_o).$$

Let $F = f + g$ such that $F(x) = f(x) + g(x)$. Then F is defined at x_o with value given as $F(x_o) = f(x_o) + g(x_o)$. Also,

$$\lim_{x \to x_o} F(x) = \lim_{x \to x_o} (f(x) + g(x)) \quad \text{exists.}$$

Finally, it is clear that

$$\begin{aligned} \lim_{x \to x_o} F(x) &= \lim_{x \to x_o} (f(x) + g(x)) = \lim_{x \to x_o} f(x) + \lim_{x \to x_o} g(x) \\ &= f(x_o) + g(x_o) = F(x_o). \end{aligned}$$

Hence, $\lim_{x \to x_o} F(x) = F(x_o)$ implies that $F = f + g$ is continuous at x_o.

□

Theorem 1.5.3. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ and $u : Z \subseteq \mathbb{R} \longrightarrow I$ be continuous functions in intervals I and Z respectively. Then, the composition of f and u defined by $f(u(x))$ is continuous in Z.*

Proof. Let the function F be f *composite* u defined by $F(x) = f(u(x))$, we show that F is continuous in Z. It suffices to show that F is defined at every point in Z and $\lim_{\Delta x \to 0} \Delta F = 0$.

Since f and u are continuous in I and Z respectively then writing $\Delta f = f(x + \Delta x) - f(x)$ and $\Delta u = u(x + \Delta x) - u(x)$ we have

$$\lim_{\Delta x \to 0} \Delta f = 0 \quad \text{and} \quad \lim_{\Delta x \to 0} \Delta u = 0.$$

Clearly, $u(x + \Delta x) = u + \Delta u$, and $\Delta u \to 0$ as $\Delta x \to 0$, so that

$$\begin{aligned} \lim_{\Delta x \to 0} \Delta F &= \lim_{\Delta x \to 0} (F(x + \Delta x) - F(x)) = \lim_{\Delta x \to 0} (f(u(x + \Delta x)) - f(u(x))) \\ &= \lim_{\Delta u \to 0} (f(u + \Delta u) - f(u)) = \lim_{\Delta u \to 0} \Delta f = 0. \end{aligned}$$

Hence, the composition of f and u defined by $f(u(x))$ is continuous in Z. □

Remark 1.5.4

Suppose f is continuous at $x_o \in X$, then

(a.) for any positive integer n, we have $\lim\limits_{x \to x_o} (f(x))^n = (f(x_o))^n$;

(b.) if $f(x) \geq 0$ and n is a positive integer, we have $\lim\limits_{x \to x_o} \sqrt[n]{f(x)} = \sqrt[n]{f(x_o)}$;

(c.) if $b > 0$, we have $\lim\limits_{x \to x_o} b^{f(x)} = b^{f(x_o)}$;

(d.) if $f(x) > 0$ and $b > 0$, we have $\lim\limits_{x \to x_o} \log_b f(x) = \log_b f(x_o)$.

We shall only state the next two important theorems and leave the proofs as exercises.

Theorem 1.5.4. *Let f be continuous on the closed interval $a \leq x \leq b$; then*

(a.) there exists a positive constant M such that $|f(x)| < M$ in $a \leq x \leq b$.

(b.) f has a minimum and a maximum value in $a \leq x \leq b$.

Theorem (1.5.4) means that if f is continuous in the *closed* interval $a \leq x \leq b$ then f is bounded and has both minimum and maximum values. For example the function f defined by $f(x) = 1/x$ is continuous in $0 < x < 1$, but possesses neither maximum nor minimum, and is not bounded in the interval $0 < x < 1$, whereas in the interval $1 \leq x \leq 2$, $f(x) = 1/x$ is bounded, possesses minimum value $f(2) = 1/2$ and maximum value $f(1) = 1$.

Theorem 1.5.5 (Intermediate Value Theorem). *Suppose the real valued function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous in $[a, b]$, then for any $\alpha, \beta \in [a, b]$ with $\alpha < \beta$, there exists x_o in the interval $\alpha < x < \beta$ such that $f(x_o)$ lies between $f(\alpha)$ and $f(\beta)$.*

Remark 1.5.5

A simple consequence of the intermediate value theorem is that, if f is continuous in $[a, b]$ and $f(a)f(b) < 0$, then there exists x_o in $a \leq x \leq b$ such that $f(x_o) = 0$. This statement implies that if f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have different signs then the equation $f(x) = 0$ has at least a solution in $[a, b]$.

Note that the intermediate value theorem is a sufficient condition for a solution of $f(x) = 0$ to exists in $[a, b]$. The theorem does not provide necessary condition for the solution of $f(x) = 0$ to exist in $[a, b]$.

Example 1.34

Show that the equation $x^3 - 3x + 1 = 0$ has at least a solution in the interval $[1, 2]$.

Solution

Let $f(x) = x^3 - 3x + 1$, then since f is a polynomial function, it is continuous in $1 \leq x \leq 2$. Also,

$$f(1) = 1 - 3 + 1 = -1 < 0 \quad \text{and} \quad f(2) = 8 - 6 + 1 = 3 > 0.$$

This implies

$$f(1)f(2) = -3 < 0.$$

Hence, $x^3 - 3x + 1 = 0$ has at least a solution in $[1, 2]$.

1.5.5 Uniform Continuity

If the function $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in an interval X, then for any given point $x_o \in X$ and any $\epsilon > 0$, we can always find $\delta > 0$ which will depend on ϵ and *possibly* x_o such that

$$|f(x) - f(x_o)| < \epsilon \quad \text{for all } x \quad \text{satisfying} \quad |x - x_o| < \delta.$$

If δ only depends on ϵ and not on any point $x_o \in X$, then f is said to be *uniformly continuous* in X.

Definition 1.5.4 (Uniform Continuity). *A real-valued function $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be uniformly continuous in an interval X if for any two points $x_1, x_2 \in X$, and given any $\epsilon > 0$, we can find a positive number $\delta = \delta(\epsilon) > 0$ which depends on ϵ only, such that*

$$|f(x_1) - f(x_2)| < \epsilon \quad \textit{whenever} \quad |x_1 - x_2| < \delta.$$

Remark 1.5.6

If a function is uniformly continuous in an interval X, then it implies that f is continuous in X. However, if f is continuous in X, it does not imply f is uniformly continuous in X.

1.5.6 Conditions for Uniform Continuity

If a function f is uniformly continuous in the open interval say $X = (a, b) = \{x \in \mathbb{R} : a < x < b\}$, then f satisfies the following necessary and sufficient conditions:

(a.) f is defined for every point of (a, b);

(b.) $\lim_{\Delta x \to 0} \Delta f = 0$ for every $x \in (a, b)$; and

(c.) $f(x)$ is finite as $x \to a^+$ and $x \to b^-$ i.e. $|f(x)|$ does not grow without bound as x approaches a from the right and b from the left.

If any of these conditions is not satisfied, then f is not uniformly continuous in the interval (a, b).
Following the conditions (a.) to (c.), we state an important theorem without proof on the uniform continuity of a function:

Theorem 1.5.6. *If a real-valued function $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in the closed interval $[a, b]$, then it is uniformly continuous in $[a, b]$.*

Example 1.35

Investigate the uniform continuity of the function f defined by $f(x) = \dfrac{1}{x}$ in the interval $0 < x < 1$.

Solution

$f(x) = \dfrac{1}{x}$ is defined for every point in $0 < x < 1$. Also, in this interval

$$\lim_{\Delta x \to 0} \Delta f = \lim_{\Delta x \to 0} (f(x + \Delta x) - f(x)) = \lim_{\Delta x \to 0} \left(\frac{1}{x + \Delta x} - \frac{1}{x} \right)$$
$$= \lim_{\Delta x \to 0} \left(-\frac{\Delta x}{(x + \Delta x)x} \right) = 0.$$

Notice what happens as $x \to 0^+$,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{1}{x} \right) = +\infty \text{ (not finite!)}.$$

Hence, f defined by $f(x) = \dfrac{1}{x}$ is not uniformly continuous in $0 < x < 1$.

Example 1.36

Investigate the uniform continuity of the function f defined by $f(x) = x^2 + 1$ in
(i.) $0 < x < 1$ (ii.) $0 < x < \infty$ (iii.) $0 < x < a$ for any $a > 0$.

Solutions

The function f defined in the given intervals satisfies conditions (a.) and (b.). Also,

$$\begin{aligned}\lim_{\Delta x \to 0} \Delta f &= \lim_{\Delta x \to 0} \left((x + \Delta x)^2 + 1 - (x^2 + 1)\right) \\ &= \lim_{\Delta x \to 0} \left(2x\Delta x + (\Delta x)^2\right) = 0.\end{aligned}$$

(i.) Given the interval $0 < x < 1$, it is clear that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 1) = 1 \quad \text{and}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2 \text{ (finite!)}.$$

Hence, f is uniformly continuous in $0 < x < 1$.

(ii.) Given the interval $0 < x < \infty$,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (x^2 + 1) = +\infty \text{ (not finite!)}.$$

Hence, f is not uniformly continuous in $0 < x < \infty$.

(iii.) Given the interval $0 < x < a$, for $a > 0$,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (x^2 + 1) = a^2 + 1 \text{ (finite!)}.$$

Hence, f is uniformly continuous in $0 < x < a$ for any real $a > 0$.

Control Problem 1.5.6

Prove that the function

(i.) f defined by $f(x) = \sin x$ is uniformly continuous in $\mathbb{R}$

(ii.) g defined by $g(x) = \cos x$ is uniformly continuous in $\mathbb{R}$

(iii.) h defined by $h(x) = \tan x$ is continuous but not uniformly continuous in the interval $0 \leq x < \frac{\pi}{2}$.